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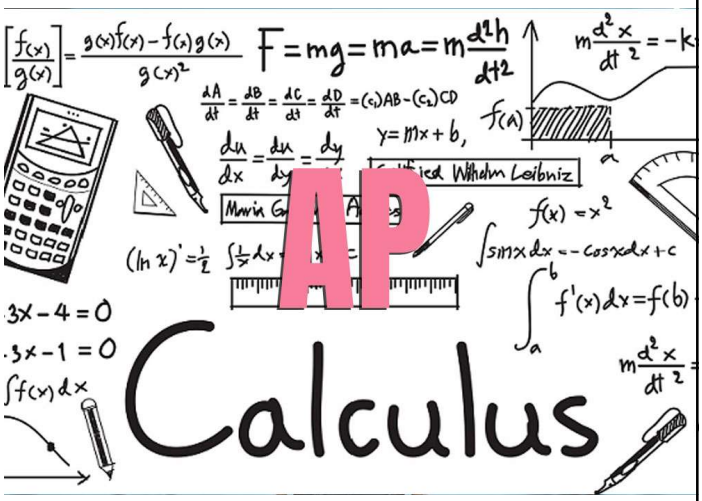
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Knowing the AP Exam!

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Historically, the free response questions on the AP Calculus AB Exam fall into these categories:

1. Data Table
2. Graph Analysis
3. Differential Equations
4. Implicit Equations
5. Particle Motion
6. Area/Volume
7. Rate In – Rate Out
8. Der. & Int. basics



Data Table: *26 #1, 25 #3, 24 #1, 23 #1, 22 #4, 21 #1, 19 #2, 18 #4, 17 #1, 16 #1, 15 #3, 14 #4, 13 #3, 12 #1* - Calculator or NO calculator

Given a table of data (usually with time and either a rate or amount function), on a closed interval with an initial amount given (typically at t=0), be able to:

a) Estimate the derivative of the given rate function (by calculating slope using the closest two points and the slope formula, (change in function) divided by (change in time).
b) Explaining the meaning of your answer from part ***a*** in the context of the problem.
c) Use estimation technique (left, right, midpoint, trapezoid) to estimate something.
d) Know how to tell if each technique is an over or underestimate (left and right by increasing/decreasing; trapezoid and midpoint by concavity).
e) Explain the meaning of the integral expression involving the function in the context of the problem. (change in amount, final amount, average value)
f) Find the average value of the function using an integral.
g) Given an equation of another function that models the function given in the table or a different function (working against the function in the table), use an integral to find some type of rate or amount.
h) Use the IVT or MVT.

2016 #1 Free Response

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Units are only graded when specified!

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

"difference quotient"

Or liters/hr/hr

$$\text{(a)}\quad R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120 \text{ liters/hr}^2$$

If $R'(3)$ then $\frac{740 - 1190}{6 - 1}$. Choose closest orderd pair NOT including the x – value in question.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\int_0^8 R(t)\,dt \approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6)$$
$$= 1(1340) + 2(1190) + 3(950) + 2(740)$$
$$= 8050 \text{ liters}$$

Ensure you specify which function is decreasing. "It" or "the function" are NOT acceptable.

This is an overestimate since R is a decreasing function.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

Important!

(c) $\text{Total} \approx 50000 + \int_0^8 W(t)\,dt - 8050$

$= 50000 + 7836.195325 - 8050 \approx 49786$ liters

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

Yes.

Therefore, the Intermediate Value Theorem guarantees at least one time t, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

Note: this was asked again on the 2026 exam but in a more leading way.

IVT: 26, 25, 22, 16
MVT: 25, 23, 19, 18, 17

2019 #2 Free Response
Particle Motion incorporated into the Data Table problem.

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	−29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

   (a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour. MVT

   (b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

   (c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

   (d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

2018 #4 Free Response
Function is NOT a rate!!

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

Related rates are almost always one part of one free response question.

Graph Analysis: $f'(x)$ *or* $g(x) = \int_2^x f(t)dt$

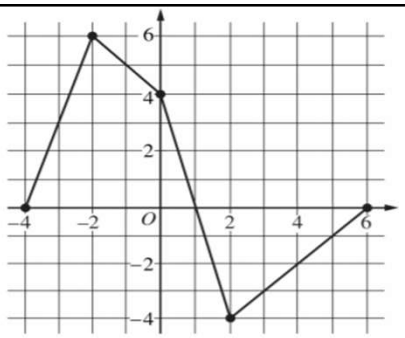
(a derivative graph stated or implied)

26 #4, 5 #4, 24 #4, 23 #4, 22 #3, 21 #4, 19 #3, 18 #3, 17 #3, 16 #3, 15 #5, 14 #3, 13 #4, 12 #3

NO calculator

a) Find values of g(x). (area with a sign…make sure that if the lower limit is larger than the upper limit that you flip the limits and take the opposite sign)
b) Determine where the relative extrema of g(x) occur. (f(x) changes signs or endpoints)
c) Find where points of inflection for g(x) occur. (f(x) changes inc. to dec. or vice-versa)
d) Find the absolute extrema of g(x) (candidate's test of critical points and endpoints).
e) Find the average rate of change of g(x). $\frac{g(b)-g(a)}{b-a}$
f) Find the tangent line at a given x-value. (point and a slope)
g) Find intervals where g(x) is increasing (f>0), decreasing (f<0), concave up (f inc.), concave down (f dec.).
h) Given a new function defined in terms of g(x) and x (as a composite function or a product of functions or a quotient of functions), find the derivative of the new function.
i) Find the limit of a function composed of g(x) and some other function. This might end up being a L'Hospital's Rule problem.

2021 #4



Graph of f

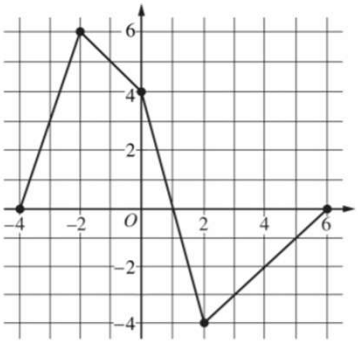
4. Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.



Graph of f

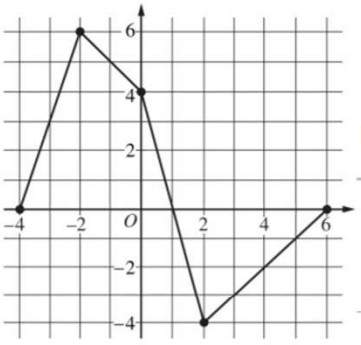
Train your students to write this connection immediately.

$G'(x) = f(x)$ in any part of the response.

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.



Graph of f

$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$

$P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$

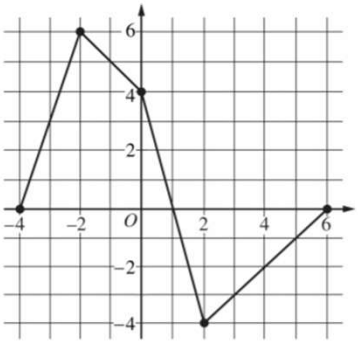
Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:

$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$

Simplification of computation is never required!

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.



Graph of f

(c) Find $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x}$.

NEVER write: $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \dfrac{0}{0}$

Acceptable: $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} \to \dfrac{0}{0}$

Numerator and denominator must be written as separate limits.

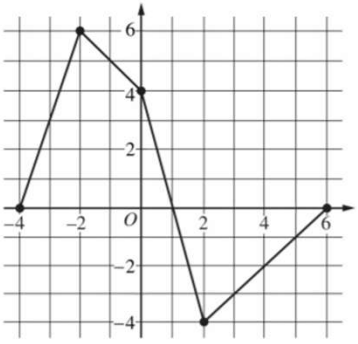
Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

$\lim_{x\to 2} G(x) = \int_0^2 f(t)\,dt = 0$.

$\lim_{x\to 2} (x^2 - 2x) = 0$

$$\lim_{x\to 2} \frac{G'(x)}{2x - 2} = \lim_{x\to 2} \frac{f(x)}{2x - 2} = \frac{f(2)}{2} = \frac{-4}{2} = -2$$

L'H: 23, 21, 19, 18



Graph of f

$G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$

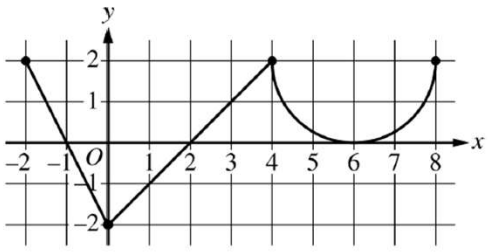
Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$

Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$.

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

2023
#4



Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

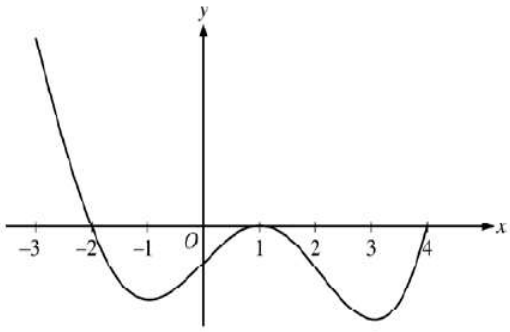
(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

2015 #5

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.



Graph of f'

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

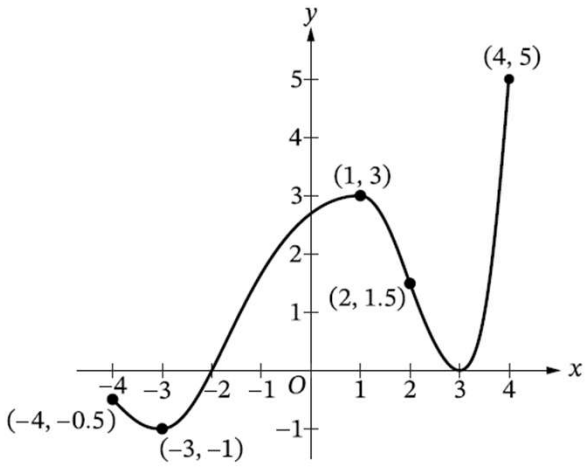
(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

AP CALCULUS AB 2026 • **FREE-RESPONSE QUESTIONS**

2026 #4

4. Let f be a twice-differentiable function on the closed interval $[-4,4]$ with $f(2)=3$. The graph of f', the derivative of f, is shown.



Graph of f'

Differential Equation:

26 #3, 24 #3, 23#2, 22 #5, 21 #6, 19 #4, 18 #6, 17 #4, 16 #4, 15 #4, 14 #6, 13 #6, 12 #5 - NO calculator

Given a differential equation, be able to:

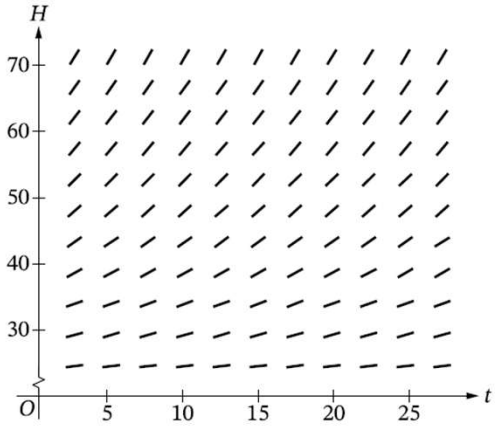
a) Know whether a given slope field represents the given differential equation.
b) Write an equation of a tangent line to solution curve and use the tangent line to approximate a nearby x-value.
c) Find the second derivative of the given differential equation.
d) Determine if a tangent line approximation is greater or less than the actual value by finding the sign of the second derivative on the interval between the point of tangency and the x-value in question (if positive, then less than; if negative, then greater than).
e) Using the second derivative, be able to answer questions about concavity.
f) Determine if a given point is a local max/min or neither. First, find the value of the first derivative by substituting the ordered pair into the differential equation. If does NOT equal zero, then neither. If DOES equal zero, then find the value of the second derivative at that point. If second derivative is positive, then local min. If negative, then local max. This is the Second Derivative test.
g) Find the particular solution (separate, integrate, solve for “C”, get “y” by itself).

2026 #3

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.



The slopes of the line segments in a slope field for the given differential equation should all be negative because $\frac{dH}{dt} < 0$ on $20 < H < 75$. However, the slopes of the line segments in the slope field given are all positive.

Give a global answer…not just at one point.

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part B

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

$$\left.\frac{dH}{dt}\right|_{(0,\,75)} = -\frac{1}{15}(75 - 20) = -\frac{11}{3}$$

2023 was very similar!

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part C

It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

For $t > 0$, $20 < H < 75$, so $\frac{d^2H}{dt^2} > 0$.

Because $\frac{d^2H}{dt^2} > 0$, the graph of H is concave up. Therefore, the tangent line lies below the graph, so the approximation is an underestimate for $H(5)$.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

$$\frac{dH}{H - 20} = -\frac{1}{15}\,dt$$

$$\int \frac{dH}{H - 20} = \int -\frac{1}{15}\,dt$$

$$\ln|H - 20| = -\frac{t}{15} + C$$

$$\ln|75 - 20| = -\frac{1}{15}(0) + C \Rightarrow \ln 55 = C$$

$$\ln|H - 20| = -\frac{t}{15} + \ln 55$$

$$|H - 20| = 55e^{-t/15}$$

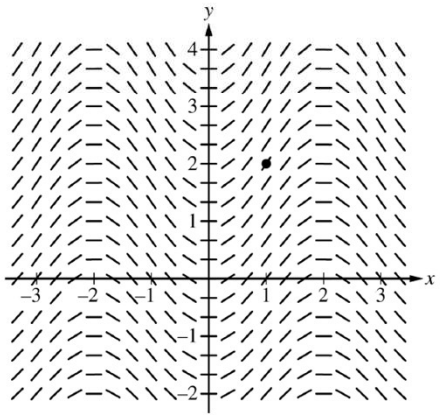
$$H - 20 = \pm 55e^{-t/15}$$

Because $H(0) = 75$, $H(t) = 20 + 55e^{-t/15}$.

2022 #5 – No context

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



Cannot be asked in free response anymore.

Implicit Equation:

25 #6, 24 #5, 23 #6, 21 #5, 15 #6 – NO calculator

Given an Implicit Equation, be able to:

a) Find the first derivative with respect to x.
b) Find points where the original curve has a horizontal tangent (y'=0: set numerator = 0)
c) Find points where the original curve has a vertical tangent (y' undefined: set denominator = 0)
d) Write an equation of a tangent line to solution curve and use the tangent line to approximate a nearby x-value.
e) Find the second derivative of the given differential equation.
f) Determine if a given point is a local max/min or neither. First, find the value of the first derivative by substituting the ordered pair into the differential equation. If does NOT equal zero, then neither. If DOES equal zero, then find the value of the second derivative at that point. If second derivative is positive, then local min. If negative, then local max. This is the Second Derivative test.
g) Find the first derivative with respect to time.

2021 #5

5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$.

y' can be used for $\frac{dy}{dx}$

$$\frac{d}{dx}\left(2y^2 - 6\right) = \frac{d}{dx}(y \sin x) \Rightarrow 4y\frac{dy}{dx} = \frac{dy}{dx}\sin x + y \cos x$$

$$\Rightarrow 4y\frac{dy}{dx} - \frac{dy}{dx}\sin x = y \cos x \Rightarrow \frac{dy}{dx}(4y - \sin x) = y \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

At the point $(0, \sqrt{3})$, $\dfrac{dy}{dx} = \dfrac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \dfrac{1}{4}$.

An equation for the tangent line is $y = \sqrt{3} + \dfrac{1}{4}x$.

Can leave in point-slope form: $y - \sqrt{3} = \dfrac{1}{4}x$ *or* $y - \sqrt{3} = \dfrac{1}{4}(x - 0)$

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

or

$\dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x} = 0 \Rightarrow y\cos x = 0$ and $4y - \sin x \neq 0$

$y\cos x = 0$ and $y > 0 \Rightarrow x = \dfrac{\pi}{2}$

When $x = \dfrac{\pi}{2}$, $y \sin x = 2y^2 - 6 \Rightarrow y \sin\dfrac{\pi}{2} = 2y^2 - 6$

$\Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0$

$\Rightarrow (2y + 3)(y - 2) = 0 \Rightarrow y = 2$

$y = \dfrac{-3}{2}$ is ignored because

When $x = \dfrac{\pi}{2}$ and $y = 2$, $4y - \sin x = 8 - 1 \neq 0$. Therefore, the line tangent to the curve is horizontal at the point $\left(\dfrac{\pi}{2}, 2\right)$.

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

$$\frac{d^2y}{dx^2} = \frac{(4y - \sin x)\left(\frac{dy}{dx}\cos x - y \sin x\right) - (y \cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$$

When $x = \frac{\pi}{2}$ and $y = 2$,

$$\frac{d^2y}{dx^2} = \frac{\left(4 \cdot 2 - \sin\frac{\pi}{2}\right)\left(0 \cdot \cos\frac{\pi}{2} - 2 \cdot \sin\frac{\pi}{2}\right) - \left(2\cos\frac{\pi}{2}\right)\left(4 \cdot 0 - \cos\frac{\pi}{2}\right)}{\left(4 \cdot 2 - \sin\frac{\pi}{2}\right)^2}$$

$$= \frac{(7)(-2) - (0)(0)}{(7)^2} = \frac{-2}{7} < 0.$$

f has a relative maximum at the point $\left(\frac{\pi}{2}, 2\right)$ because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

Motion:

26 #5, 25 #5, 24 #2, 23 #2, 22 #6, 21 #2, 19 #2 (data table), 18 #2, 17 #2, 16 #2, 14 #4, 13 #2, 12 #6 – Calculator or No calculator

Given a velocity function (and possibly a position function of a 2nd particle) be able to:

a) Determine when the particle is moving left (v<0) or right (v>0) or stopped (v=0). (Velocity equal to zero and make a sign chart or use calculator for sign of velocity.)

b) Determine when the particle changes directions (velocity=0 ***and*** changes signs).

c) Determine if the particle is speeding up (speed increasing) or slowing down (speed decreasing) and justify your answer. If velocity and acceleration at a given time are the same signs, then speeding up (speed increasing); if signs are opposite then slowing down (speed decreasing).

d) Find the position of the particle at any given time (initial position + integral of velocity).

e) Calculate total distance traveled (integral of absolute value of velocity).

f) Find the acceleration of the particle at any given time. (derivative of velocity)

g) Find average velocity: either $\frac{1}{b-a}\int_a^b v(t)dt$ ***or*** $\frac{x(b)-x(a)}{b-a}$

h) Determine if the particles are moving toward or away from each other at a given time. (compare the positions and velocities of each particle)

2021 #2

2. A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

(a) Find the positions of particles P and Q at time $t = 1$.

(b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

(c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

(d) Find the total distance traveled by particle P over the time interval $0 \le t \le \pi$.

A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

→ Y1

A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

→ Y2

(a) Find the positions of particles P and Q at time $t = 1$.

$x_P(1) = 5 + \int_0^1 v_P(t)\,dt = 5.370660$

"Integrate a rate and you get change in amount of the original function."

At time $t = 1$, the position of particle P is $x = 5.371$ (or 5.370).

$x_Q(1) = 10 + \int_0^1 v_Q(t)\,dt = 8.564355$

At time $t = 1$, the position of particle Q is $x = 8.564$.

A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

(b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

$v_P(1) = \sin\left(1^{1.5}\right) = 0.841471 > 0$

At time $t = 1$, particle P is moving to the right.

$v_Q(1) = (1 - 1.8) \cdot 1.25^1 = -1 < 0$

At time $t = 1$, particle Q is moving to the left.

At time $t = 1$, $x_P(1) < x_Q(1)$, so particle P is to the left of particle Q.

Thus, at time $t = 1$, particles P and Q are moving toward each other.

A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

(c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

$a_Q(1) = v_Q'(1) = 1.026856$

The acceleration of particle Q is 1.027 (or 1.026) at time $t = 1$.

$v_Q(1) = -1 < 0$ and $a_Q(1) > 0$

The speed of particle Q is decreasing at time $t = 1$ because the velocity and acceleration have opposite signs.

Only acceptable justification is to compare the signs of velocity and acceleration.

A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin\left(t^{1.5}\right)$ for $0 \le t \le \pi$. At time $t = 0$, particle P is at position $x = 5$.

A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \le t \le \pi$. At time $t = 0$, particle Q is at position $x = 10$.

(d) Find the total distance traveled by particle P over the time interval $0 \le t \le \pi$.

$$\int_0^{\pi} |v_P(t)|\, dt = 1.93148$$

Over the time interval $0 \le t \le \pi$, the total distance traveled by particle P is 1.931.

Area & Volume:

26 #2, 25 #2, 24 #6, 22 #2, 21 #3, 19 #5, 16 #5, 15 #2, 14 #2, 13 #5, 12 #2

Calculator or No calculator

Given a graph of a region(s) defined by two or more given functions, be able to:

a) Calculate the area of the region. (integral of upper minus lower OR integral of absolute value of one function minus the other function if more than one region)

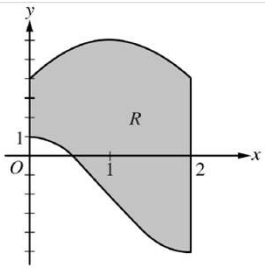
b) Calculate the volume of the solid formed by rotating the region around a given axis or another vertical or horizontal line:

$$\pi \int_a^b [(further - axisofrotation)^2 - (closer - axisofrotation)^2]\, d__$$

c) Calculate the volume of the solid formed by cross sections (squares, semi-circles, isosceles right triangles, rectangles, or an unknown shape with the area formula given) across the region.

d) A follow-on question that deals with area, volume, or related rate.

2019 #5

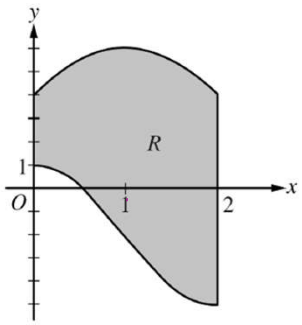


Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.



(a) Find the area of R.

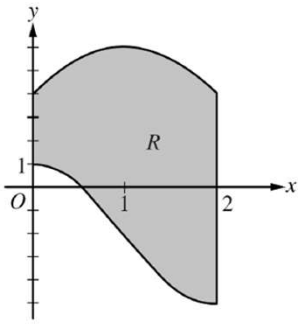
Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

$$\int_0^2 (h(x) - g(x))\,dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right) dx$$

$$= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$$

$$= \left(\left(12 - \frac{2}{3}\right) - (-4+0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0+0)\right)$$ ← I would stop here!

$$= 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$$

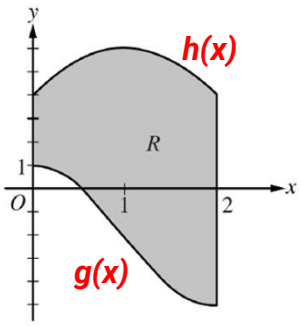


$$\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$$
$$= \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$$

The volume of the solid is $\ln 5 - \ln 3$.

Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.



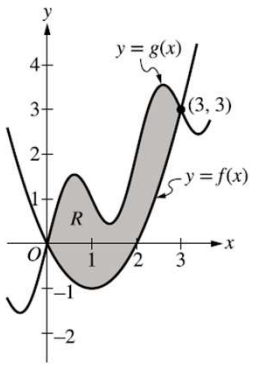
Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

(c) $\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right) dx$ OR $\pi\int_0^2 [(g(x)-6)^2 - (h(x)-6)^2]\,dx$

2025 #2

2. The shaded region R is bounded by the graphs of the functions f and g, where $f(x) = x^2 - 2x$ and $g(x) = x + \sin(\pi x)$, as shown in the figure.



(Note: Your calculator should be in radian mode.)

A. Find the area of R. Show the setup for your calculations.

B. Region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis is a rectangle with height x and base in region R. Find the volume of the solid. Show the setup for your calculations.

C. Write, but do not evaluate, an integral expression for the volume of the solid generated when the region R is rotated about the horizontal line $y = -2$.

D. It can be shown that $g'(x) = 1 + \pi\cos(\pi x)$. Find the value of x, for $0 < x < 1$, at which the line tangent to the graph of f is parallel to the line tangent to the graph of g.

Rate In/Out:

25 #1, 22 #1, 19 #1, 18 #1, 17 #2, (16 #1 in the data table), 15 #1, 14 #1

Calculator active

Given a rate function or functions (water pouring into and out of a tank, people getting onto and off an escalator, pounds of bananas being put onto a shelf and being removed from a shelf, etc.), be able to:

a) Find the amount at any given time (initial amount plus integral of rate).
b) Calculate total amount (integral of rate).
c) Find the time or amount of an absolute max or min using the Candidate's Test.
d) Determine if the rate or amount is increasing or decreasing at a given time. (compare the values of the derivatives of the amount functions or of the rate functions depending on what is being asked)
e) Find the value of the derivative of the given function and explain the meaning of the answer (with correct units) in the context of the problem.
f) Find average value of the rate.

2019 #1

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

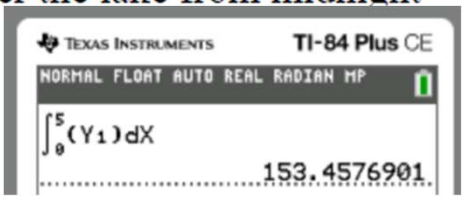
Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

(a) $\int_0^5 E(t)\,dt = 153.457690$

In calculator, enter:
E(t) → Y1
L(t) → Y2

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.



Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

Did not need the title "A(t)"

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore, the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

(d) $E'(5) - L'(5) = -10.7228 < 0$

Or: E'(5) = -6.802
L'(5) = 3.921
E'(5) < L'(5)

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

$\frac{d}{dx}(Y_1)|_{X=5}$
-6.801747307
$\frac{d}{dx}(Y_2)|_{X=5}$
3.92103316

Derivative & Integration basics:

26 #6, 23 #5, 19 #6, 18 #5, 16 #6, 14 #5
NO calculator

Given a table of x values with the y-values and derivative values of two functions (let's call them f and g) **AND/OR** be given a graph of one or both functions **AND/OR** be given the equation of a function (usually a product or quotient of two functions), be able to:

a) Find the derivative of a third function (let's call it h) that is some sort of composite of the two given functions such as: $h(x) = f(g(x))$ or $h(x) = \frac{f(x)}{g(x)}$ or $h(x) = f(x)g(x)$.
b) If you are given the signs of the values of the intervals between the ordered pairs (so that you know when the derivatives are positive and negative) identify local extrema.
c) If given the equation, find absolute extrema using Candidate's Test.
d) Find the value of some interesting integrals (like $\int_1^3 f''(x)dx = f'(x)|_1^3 = f'(3) - f'(1)$
e) Find average rate of change.
f) Find the equation of a tangent line.
g) Use L'Hospital's Rule to find a limit.
h) Use MVT or IVT.
i) Use of the Squeeze Theorem (2019 is the only year I have seen it tested and that part of the problem was worth 1 point.)

2018 #5

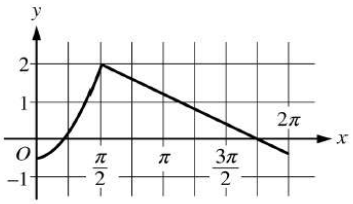
5. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

The average rate of change of f on the interval $0 \le x \le \pi$ is

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-e^{\pi} - 1}{\pi}.$$

Let f be the function defined by $f(x) = e^x \cos x$.

(b) What is the slope of the line tangent to the graph of f at $x = \dfrac{3\pi}{2}$?

$$f'(x) = e^x \cos x - e^x \sin x$$

$$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2}\cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2}\sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$$

The slope of the line tangent to the graph of f at $x = \dfrac{3\pi}{2}$ is $e^{3\pi/2}$.

Let f be the function defined by $f(x) = e^x \cos x$.

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \dfrac{\pi}{4}, x = \dfrac{5\pi}{4}$

x	$f(x)$
0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}e^{\pi/4}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}e^{5\pi/4}$
2π	$e^{2\pi}$

The absolute minimum value of f on $0 \le x \le 2\pi$ is $-\dfrac{1}{\sqrt{2}}e^{5\pi/4}$.

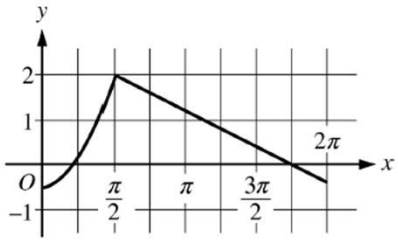
Options for declaring final answer:
1. Circle the function value.
2. Write the function value.
3. If unsure, write:

$$\min \text{ of } \frac{-1}{\sqrt{2}}e^{5\pi/4} \text{ at } x = \frac{5\pi}{4}$$

* CANNOT just write or circle the ordered pair.

Let f be the function defined by $f(x) = e^x \cos x$.

Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

$\lim_{x \to \pi/2} f(x) = 0$

Because g is differentiable, g is continuous.

$\lim_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$

By L'Hospital's Rule,

$$\lim_{x \to \pi/2} \frac{f(x)}{g(x)} = \lim_{x \to \pi/2} \frac{f'(x)}{g'(x)} = \frac{-e^{\pi/2}}{2}.$$

2026 #6

6. The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	0	2	3	6
$f(x)$	−1	3	8	5
$f'(x)$	−5	4	9	−2

Part A

Find $\lim_{x \to 2} \frac{f(x)}{x}$, or state that the limit does not exist.

Part B

Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

Part C

Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

Part D

Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$.

i. Find $k'(x)$.

ii. Find $k''(3)$. Show the work that leads to your answer.

2016 #6

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	−6	3	2	8
2	2	−2	−3	0
3	8	7	6	2
6	4	5	3	−1

The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \frac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

Related Rates are typically tested as one part of one free response question each year.

2025 #6 – Implicit
2024 #5 – Implicit
2023 #6 – Implicit
2022 #2 – Volume
2022 #4 – Data Table
2019 #4 – Differential Equation
2018 #4 – Data Table
2017 #1 – Data Table
2016 #5 – Volume

1. RADIAN mode!!!

2. Important required skills:
   - Graphing in an appropriate viewing window
   - Finding zeros of functions
   - Solving equations
   - Calculating the derivative of a function at a value
   - Calculating the value of a definite integral

3. Show the mathematical equation that leads to the calculator result.

4. Round answers to ***three decimal places.***
   ***Use four decimal places if you are using that number for further calculations.

5. Use the fraction key to avoid parenthesis errors.

6. Maintain accuracy.
   Use CALC INTERSECTION rather than TRACE.
   Use the HOME screen for calculations rather than the GRAPH screen.

7. Justifications require mathematical (non-calculator) justifications.
   A calculator graph is not a sufficient justification on a FRQ.

8. The calculator sometimes has a hard time communicating...
   * Calculator might say "4.8372E-26" when it is trying to say "0"
   * Calculator will not graph vertical or steep graphs...edges of graphs that look like they are floating above the x-axis might actually go to the x-axis (like a semi-circle).

If using the TI, it would be very helpful to know:

- Shortcut key for inputting ***fractions*** (Alpha F1)
- Shortcut key for the ***y-vars*** (Alpha F4) menu.
- How to graph the derivative of a function by entering the function in Y1, and then in Y2 use the MATH 8 nDeriv function with x=x (and if you need the second derivative use nDeriv again in Y3).
- How to store values to maintain accuracy.