2025 BC-6 Chuck Garner, Rockdale Magnet School

Problem Overview

$$\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2\cdot 3} + \frac{(x-4)^3}{3\cdot 3^2} + \frac{(x-4)^4}{4\cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$$

Students were given that the Taylor series for a function f centered at $x = 4$ is given by the series above. Students were also told that the series converges to f in its interval of convergence.

Part a

Students were asked to find the interval of convergence of the given series using the ratio test, and to justify their answer.

Part b

Students were asked to write the first three terms and the general term of f' about $x = 4$.

Part c

Students were told that the series in part (b) is a geometric series. Students were asked to show, for all real numbers in the interval of convergence of f', that $f'(x) = (x-4)/(7-x)$.

Part d

Students were told that the radii of convergence of f about $x = 4$ and of f' about $x = 4$ are the same. Students were asked to determine whether the series for $f'(x)$ in part (b) converges to $(x-4)/(7-x)$ at $x = 8$ and to give a reason for their answer.

Comments on Student Responses and Scoring Guidelines

Part a

The first part could earn the student five points, P1 through P5.

P1 was earned for presenting a correct ratio, with or without absolute value, with or without a limit, with or without simplifying. Any one of the following earned the point:

$$\frac{\dfrac{(x-4)^{n+2}}{(n+2)3^{n+1}}}{\dfrac{(x-4)^{n+1}}{(n+1)3^n}}, \qquad \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}, \qquad \frac{(x-4)^n(x-4)^2(n+1)3^n}{3(n+2)3^n(x-4)^n(x-4)}, \quad \text{or} \quad \frac{(n+1)(x+4)}{3(n+2)}.$$

To earn P1, students had to enter the problem with one of these ratios. P1 was also earned for presenting the reciprocal of the correct ratios, and such a response remained eligible for P2, P3, and P4, but did not earn P5.

P2 was earned for presenting the limit of the absolute value of the student's ratio and correctly evaluating the limit. Simplifying the ratio need not be shown, but must be correct if shown. The correct limit is

$$\lim_{n\to\infty}\left|\frac{(x-4)(n+1)}{3(n+2)}\right| = \left|\frac{x-4}{3}\right|.$$

P2 could be earned without using absolute value provided the student resolved the error when writing the interval. That is, a student presenting the above work but without the absolute value could earn this point if the student went on to write $-3 < x-4 < 3$, or an equivalent interval. A response that evaluated the limit incorrectly did not earn P2, but Readers were allowed to read with the student for P3 and P4.

P3 was earned for presenting a correct finite interval of convergence centered at $x = 4$. The correct interior of the interval of convergence is $1 < x < 7$. Any finite interval with a center of $x = 4$ resulting from the evaluation of the student's limit earned this point. A student could enter the problem by writing $|(x-4)/3| < 1$, which earned P3, but the student doing this did not earn P1 nor P2.

P4 was earned for considering both endpoints of the interval $1 < x < 7$. P4 was also earned for considering both endpoints of any other finite interval that earned P3.

P5 was earned for testing both endpoints for convergence and writing the final interval of convergence. When $x = 1$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}3}{n+1}$$

which converges by the alternating series test. When $x = 7$, the series becomes

$$\sum_{n=1}^{\infty}\frac{3}{n+1}$$

which diverges by comparison to the harmonic series, or by the integral test. The response could include only the name of the tests used, or present the work using the tests, or both (but if a comparison test is used, the series used in the comparison must be presented). Thus, the interval of convergence is $1 \le x < 7$. It did not matter how the student expressed this interval; as the inequality or as $[1, 7)$.

If a student made a typo in the ratio, such as writing $x + 4$ instead of $x - 4$, we read with the student. The response did not earn P1 nor P5, but was eligible for P2, P3, P4, and P9. If the typo resulted in the evaluation of a limit of 0, and therefore an interval of all real numbers, we again read with the student. The response did not earn P1, P4, nor P5, but was eligible for P2, P3, and P9.

Part b

This part of the question could earn the student points P6 and P7.

P6 was earned for presenting the first three terms of the Taylor series, and P7 was earned only for the correct general term. No work needed to be simplified, but must be correct if simplified. The simplified terms come directly from differentiating the given series:

$$f'(x) = \frac{x-4}{3} + \frac{(x-4)^2}{3^2} + \frac{(x-4)^3}{3^3} + \cdots + \frac{(x-4)^n}{3^n} + \cdots .$$

These terms may be presented as a polynomial, a series, or a list.

If more than the first three terms and the general term were presented, they were not read.

Part c

Only P8 was available to the student in this part.

Students need to show, using the sum of an infinite geometric series, that f' is equivalent to $(x-4)/(7-x)$. Students should use the formula for the sum of infinite geometric series with first term $(x-4)/3$ and common ratio $(x-4)/3$. Ideally, this work is presented as

$$\frac{(x-4)/3}{1-(x-4)/3} = \frac{x-4}{3-(x-4)} = \frac{x-4}{7-x}.$$

Unsimplified correct answers earned P8. Any errors in arithmetic or simplification did not earn P8. Other work, correct or incorrect, presented by the student in this part was not read.

Part d

In the last part, P9 was available to the student.

Students needed to give a reason consistent with the interval of convergence they obtained in part (a), even if that interval was incorrect. A minimal solution earning P9, in the presence of nothing else to contradict it, is "No, because $x = 8$ lies outside the interval of convergence."

Many students substituted $x = 8$ into the series for $f'(x)$. This made the series a geometric series with common ratio $4/3$. The responses that presented the geometric series and argued that $4/3 > 1$ and so the geometric series diverges, earned P9. However, they had to explcitly show that the common ratio was greater than 1, but this could be presented many ways. One way that earned P9 was

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n \qquad r > 1 \quad \Rightarrow \quad \text{diverges.}$$

We read the unlabled r as $4/3$. However, simply saying $r > 1$ without presenting a ratio of $4/3$ somewhere in the work for part (d) did not earn P9.

Some responses presented the geometric series and then used the nth term test to show divergence. This also earned P9.

Observations and Recommendations for Teachers

(1) The vast majority of students earned P1 by setting up the correct ratio. However, many of these students did not earn P2 because their response did not indicate the evaluation of a limit. In many responses, the letters "lim" were absent. Without this, there is no confirmation that the student is actually using the ratio test. Students should be careful in indicating the work they intend to show.

(2) Another reason responses did not earn P2 was that the students simply dropped the constant 3 from their ratio. This Reader saw responses such as

$$\lim_{n\to\infty}\left|\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}\cdot\frac{(x-4)^{n+1}}{(n+1)3^{n}}\right| = \lim_{n\to\infty}\left|\frac{(x-4)(n+1)}{n+2}\right| = |x-4| < 1$$

which leads to the interval $3 < x < 5$ (which is eligible for P3 and P4). A startling number of responses simply could not add 4 to -3 correctly: from $-3 < x - 4 < 3$ students wrote $-1 < x < 7$. This did not earn P3, and since this is no longer centered at $x = 4$, locked them out of earning P4 and P5 as well. Poor notation hindered some students earning points. Consider this response, seen often as students were evaluating the limit:

$$\lim_{n\to\infty}\left|\frac{(x-4)(n+1)}{3(n+2)}\right| = |x-4|\lim_{n\to\infty}\left|\frac{n+1}{3(n+2)}\right| = \frac{1}{3}.$$

What the student did with this next determined whether the student recovered and earned P2, or was lost and did not earn P2 (or P3, P4, and P5). Students should take a few seconds to check their work.

(3) Many students did not earn P4 nor P5 because they never checked the endpoints for convergence. This is an expected skill on the BC exam, and students should have opportunities to practice this.

(4) When it comes to P5, this Reader saw many attempts to justify the convergence or divergence of the endpoints. The following response is the ideal minimal response that earns P5.

"$x = 1$: $\sum \dfrac{(-1)^n 3}{n+1}$ converges by AST. $\quad x = 7$: $\sum \dfrac{3}{n+1}$ diverges by LCT with $\dfrac{1}{n}$."

We do not like the fact that the student leaves off the summation index, and we really don't like that the student compares the series to the *sequence* $1/n$. However, the student names the correct tests and indicates enough of the comparison series to earn the point. With only one point to award for all of this work, responses deemed just barely adequate were awarded the point. The following response, seen by many Readers, did not earn the point due to the wrong justification for the endpoint $x = 7$.

$$x = 7\text{: ``}\frac{3}{n+1} \Rightarrow \text{harmonic} \Rightarrow \text{diverges.''}$$

Unfortunately, the series $3/(n+1)$ is *not* the harmonic series. The harmonic series is defined to be only the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The series $\sum 3/(n+1)$ *behaves like* the harmonic series but it is not the harmonic series. Students should be able to use the harmonc series as as a standard known series so that the limit or direct comparison tests can be used effectively. Another response seen often that did not earn P5 is this one:

$$x = 7\text{: ``}\frac{3}{n+1} \text{ diverges by } p\text{-series.''}$$

Unfortunately, this is not a p-series either! A p-series is defined to be only a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

not a series of the form $\sum 1/(n^p + k)$, where k is a constant. Again, such a series may *behave* like a p-series, but that means we may compare it to a p-series (or use the integral test). Students should be able to use p-series as a tool for a comparison test.

(5) In part (b), many students tried to construct a Taylor series using the "formula" for a Taylor series. Of course this was not necessary, and these responses did not earn P6 nor P7. Students should be taught how to manipulate a series representing a function in order to find the series for the derivative or the antiderivative.

(6) About a fourth of the responses this Reader saw (out of thousands) never bothered to find the general term in part (b), even though the first three terms were presented correctly. The only assumption is that students did not read the question. Students should read the problem and do what is asked.

(7) Once again, notation issues resulted in points not being earned. Responses did not earn P7 because the general term was expressed as

$$\frac{n+1(x-4)^n}{(n+1)3^n} \qquad \text{instead of} \qquad \frac{(n+1)(x-4)^n}{(n+1)3^n}.$$

If the response included a subsequent simplified expression, such as

$$\frac{n+1(x-4)^n}{(n+1)3^n} = \frac{(x-4)^n}{3^n},$$

the student is said to have "recovered" and the point was earned. Students should be careful to use parentheses to avoid ambiguous expressions.

(8) In part (c), the point was earned for showing the verification that the series for f' can be expressed as $(x-4)/(7-x)$. Many students wrote work that had nothing to do with this verification, and simply presented at the end of the work "$f'(x) = (x-4)/(7-x)$" which did not earn the point. Other students wrote

the formula for the sum of a geometric series (first term over 1 minus the common ratio), and attempted to show the verification. Many responses went like this:

$$\text{“}\frac{a_1}{1-r} = \frac{1}{1-\dfrac{x-4}{3}} = \frac{x-4}{3} \times \frac{3}{3-(x-4)} = \frac{x-4}{7-x}.\text{”}$$

This looks like the student shows the verification; however, the student has explicitly stated that the first term is 1, not $(x-4)/3$, by linking the first two expressions with an equals. Then a multiplication by $(x-4)/3$ in the third expression appears from nowhere. This did not earn the point. However, the presentation of the following did earn P8:

$$\text{“}\frac{1}{1-r} = \frac{1}{1-\dfrac{x-4}{3}} = \frac{3}{7-x} \qquad \frac{x-4}{3} \times \frac{3}{7-x} = \frac{x-4}{7-x}.\text{”}$$

This response calculates the infinite geometric sum without considering the first term, and then multiplies that by the first term, since $a_1/(1-r)$ is the same as $a_1 \times 1/(r-1)$. This earned P8. It is generous that students were given the answer that they should work towards, but as a result, earning P8 became all about the work. Without the correct simplification starting from the sum of an infinite geometric series with first term and common ratio both $(x-4)/3$, the response did not earn the point. Like with any verification problem on the AP Exam (including verifying the derivative of an implicit function), the work matters. Students should be given some practice problems in verifying an answer so they are allowed to consider the work as the answer, and not “the answer” as the answer.

(9) In part (d), there were many responses with a justification based on the radius of convergence: “No, $x = 8$ is outside the radius of convergence.” However, the radius is a length, and without knowing where the radius is, one cannot know what lies “outside” of it. Such a response did not earn P9. However, if the response was a little more detailed, P9 was earned: “No, since the radius is 3 and the center is 4, $x = 8$ is outside the radius of convergence.” With reference to the center (which, along with the radius, indicates an interval), the point was awarded. Students must justify convergence with respect to an interval; the radius is a tool used, with the center, to find the interval.

(10) Again in part (d), students used phrases such as “$x = 8$ is not in the range of $1 < x < 7$,” “$x = 8$ is beyond the limits of $1 < x < 7$,” and “$x = 8$ is out of bounds of $1 < x < 7$.” These were accepted for P9 with explicit use of the interval. That is, the response “$x = 8$ is not in the range” without an interval presented did not earn the point. Students should use the proper calculus vocabulary to effectively communicate their reasoning.

(11) Many students reasoned that because the substitution of $x = 8$ in the series for $f'(x)$ and $x = 8$ into the expression $(x-4)/(7-x)$ gave different results, the series diverged at $x = 8$. This reasoning did not earn P9 because the differing values is a *consequence* of the divergence of the series at $x = 8$, not the *reason* for the divergence at $x = 8$.