2025 BC-5 Chuck Garner, Rockdale Magnet School

Problem Overview

Students were given that $y = f(x)$ is the solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$, with $f(1) = -1$.

Part a

Students were asked to find $f''(1)$ and to show their work.

Part b

Students were asked to write the second-degree Taylor polynomial for f centered at $x = 1$.

Part c

Students were told that $|f'''(x)| \leq 60$ in the interval $[1, 1.1]$, and then asked to use the Lagrange error bound to show that the difference between the approximation of $f(1.1)$ by the second-degree Taylor polynomial for f at $x = 1$ and the actual value of $f(1.1)$ is at most 0.01.

Part d

Students were asked to use Euler's method with two steps of equal size to approximate $f(1.4)$, starting at $x = 1$, and to show their work.

Comments on Student Responses and Scoring Guidelines

Part a

The first part could earn the student P1, P2, and P3. The correct differentiation of the equation is

$$\frac{d^2y}{dx^2} = -y^2 + (3 - x)(2y)\frac{dy}{dx}.$$

Correct use of the product rule earns P1, and correct use of the chain rule earns P2. Using only the product rule without the chain rule earned P1 but rendered the student ineligible for P2 and P3. A response that did not earn P1, but used the chain rule correctly, i.e.,

$$\frac{d^2y}{dx^2} = -2y\frac{dy}{dx}, \qquad \text{or} \qquad \frac{d^2y}{dx^2} = 6y\frac{dy}{dx} - 2y\frac{dy}{dx},$$

earns P2 and is eligible for P3 with a consistent answer. Any algebraic errors in simplifying a correct derivative made the response ineligible for P3.

To earn P3, students must present the value, simplified or unsimplified, of $f''(1) = -9$ (or a consistent value from a derivative that earned P2). Any errors in simplifying a correctly presented expression $-(-1)^2 + (3-1)(2)(-1)(2)$ still earned P3. In fact, presenting $-(-1)^2 + (3-1)(2)(-1)(2)$ earned P1, P2, and P3.

Students who immediately separated the variables and solved the differential equation were eligible for P1, P2, and P3 provided that they used the initial value and that the derivative of their solution $y = f(x)$ was equivalent to $(3-x)y^2$. However, the response was not eligible for points until the response presented a second derivative.

Part b

This part of the question could earn the student points P4 and P5. The Taylor polynomial is

$$P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2.$$

Presenting two correct terms of the polynomial earned P4, and presenting the third term and writing their response as a second-degree polynomial earned P5. However, the response must present the above polynomial before any simplification to earn P4 and P5; indeed, a response that only presents a simplified polynomial, such as $-\frac{15}{2} + 11x - \frac{9}{2}x^2$, earns neither P4 nor P5. However, any subsequent simplification errors of the correct polynomial above still earned the response P4 and P5. A response that attached additional terms to the polynomial or included $+\cdots$ did not earn P5. A response importing incorrect values of $f'(1)$ and $f''(1)$ from part (a) was still eligible for P4 and P5.

Part c

Both P6 and P7 were available in this part. By the Lagrange error bound, we have that

$$|f(1.1) - P_2(1.1)| \le \frac{\max\limits_{1\le x\le 1.1} |f'''(x)|}{3!}|1.1-1|^3 = \frac{60}{6}(0.1)^3 = 10\cdot\frac{1}{1000} = \frac{1}{100} = 0.01.$$

Presenting either

$$\frac{\max\limits_{1\le x\le 1.1} |f'''(x)|}{3!}|1.1-1|^3 \qquad \text{or} \qquad \frac{60}{6}(0.1)^3$$

earns P6, and subsequent simplification errors will not earn P7. To earn P7, the response must earn P6 and explicitly connect the error bound with 0.01. This connection could be presented as "error ≤ 0.01," "error bound $= 0.01$," or any equivalent statement. However, "error $= 0.01$" and "error < 0.01" are incorrect and neither earns P7.

Part d

In the last part, points P8 and P9 were available. An correct first step of Euler's method must be presented to earn P8. Such a first step can be presented as $-1 + 0.2(2)$ or $-1 + (1.2-1)(3-1)(1^2)$. An incorrect

value of the derivative can be imported from part (a) and still earn P8 (and P9). Any simplification errors of this first step render the response ineligible for P9. To earn P9, the response must present the value $-0.6 + 0.2(3 - 1.2)(-0.6)^2$ or a simplified equivalent, with supporting work. Subsequent errors in simplification of the correct value still earn P9.

If the steps of Euler's method are presented in a table, the table does not need to be labeled to earn P8 and P9, provided the values are correct. In the presence of no answer or an incorrect answer (which does not earn P9), the table must be labeled in order to earn P8.

Observations and Recommendations for Teachers

(1) Reading is important. Students who solved the differential equation in part (a) did much more work than they needed to. Clearly, these students saw a differential equation and believed it needed to be solved, even though the problem never asked them to. Had they actually read the problem (instead of relying on their memory of always solving them in class), they could have been more productive in other problems on the exam. Teachers should give students some differential equations problem like this one, in which the solution is never required. See, for example, AB4 on the 2015 AP Calculus Exam.

(2) Attempts to differentiate dy/dx were riddled with interesting errors: students indicated that the derivative of $3 - x$ is $-1x$, 3, or 1, and students differentiating $(3 - x)y^2$ as $-2y$. Some students distributed before differentiating, writing $dy/dx = 3y^2 - xy^2$, and then didn't realize that the second term requires the product rule. Some students simply attached dy/dx to every term of their second derivative. Students should have practice differentiating implicitly in which more than one differentiation rule is required.

(3) Some responses centered the Taylor polynomial at $x = 0$, which did not earn P4 or P5. Students should be given practice with Taylor polynomials centered at points other than zero.

(4) Two of the biggest issues in part (c) were 1) the lack of communicating correctly that the student was using the maximum value of the third derivative, and 2) poor communication of that the error is less than or equal to 0.01. In the first case, in the presence of the correct answer, P6 was earned regardless of this poor communication. In the second case, poor communication may cost the student P7. Notation is important, and correctly using good notation should be stressed in class.