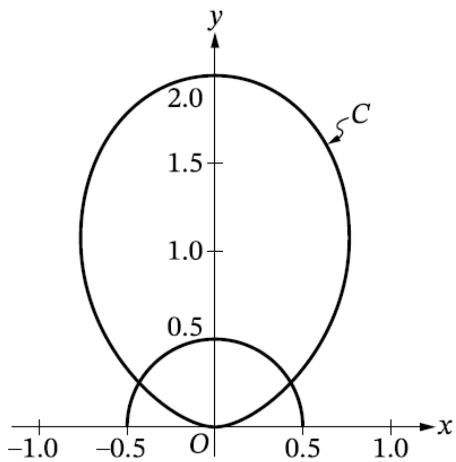
 Kimberley Thomas, Moon Valley High School

Problem Overview:

Students were given the graph below of two polar curves showing curve C defined by $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$ and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$. Students were reminded that calculators should be in radian mode.



Part a:

Students were asked to show the set up and find the rate of change r with respect to θ at a point on curve C where at $\theta = 1.3$.

Part b:

Students were asked to show the set of the calculations and find the area of the region that lies inside curve C and outside of the semicircle.

Part c:

Students were given $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C and asked to find the value of θ that corresponds with the point on curve C that is farthest from the y-axis on the interval $0 \le \theta \le \frac{\pi}{2}$. Students were reminded to justify their answer.

Part d:

Students were given that a particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Students were asked to show the set up for their calculations and find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$.

Comments on student responses and scoring guidelines:

Part a: worth 1 point

Differentiation of r at $\theta = 1.3$ was needed to arrive at the correct answer. **P1** was earned for indicating the differentiation of r and the correct answer. The correct answer could be exact, rounded to three decimal places, or truncated. $\left.\frac{dr}{d\theta}\right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3) = 1.031003$. Miscommunications or linkage errors were not penalized. For example, $r' = 1.031$, $r'(\theta) = 1.031$, and $\frac{dr}{d\theta} = 1.031$ were sufficient to earn **P1** despite the linkage error.

Part b: worth 3 points

To earn **P2**, students must present a definite integral including $(r(\theta))^2$ with or without the differential $d\theta$. **P3** is earned for a definite integral or integrals that present the correct integrand, $(r(\theta))^2 - \left(\frac{1}{2}\right)^2$ with or without the differential $d\theta$. The limits of integration, $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{5\pi}{6}$, and the factor $\frac{1}{2}$ are addressed in **P4**. **P4** is earned for the correct answer presented as an exact answer, rounded or truncated to 3 decimal places with or without supporting work. Using symmetry of the region correctly earns **P2**, **P3**, and **P4**. Incorrect or unclear communication between the correct integral and correct answer is treated as scratch work and not considered for scoring. For example, the linkage is not considered for $\int_{\pi/6}^{5\pi/6}\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right) d\theta = 2.067$, which earned **P2** and **P3** for the integral and **P4** for the correct answer. An indefinite integral does not earn **P2**, earns **P3** with the correct integrand, and is eligible for **P4** with the correct answer.

Part c: worth 3 points

Considering $\frac{dx}{d\theta} = 0$ earned **P5**. The answer $\theta = 0.955317$ was not sufficient to earn **P5**. Statements discussing $\frac{dx}{d\theta}$ changing signs or using the phrase "critical point of $x(\theta)$" also earn **P5**. **P6** is the justification point and is earned for correctly presenting the candidates test or a correct global argument. For the candidates test, **P6** is earned for correctly evaluating $x(\theta)$ at $\theta = 0$, $\theta = 0.955317$, and $\theta = \frac{\pi}{2}$. Values must be correct to the first digit after the decimal (rounded or truncated). **P6** is not earned for a local argument, but is eligible for **P7**. An example of a global argument is: Because $\theta = 0.955317$ is the only critical point on the interval $0 \le \theta \le \frac{\pi}{2}$ and $\frac{dx}{d\theta}$ changes sign from positive to negative at $\theta = 0.955317$, thus $\theta = 0.955317$ is the location of the absolute maximum for $x(\theta)$ on the interval. **P7** is earned for the correct answer $\theta = 0.955317$ with supporting work. Presentation of a local argument or an incorrect global argument is still eligible for **P7**.

Part d: worth 2 points

P8 is earned for symbolic or numeric presentation of $\frac{dr}{d\theta} \bullet \frac{d\theta}{dt}$. **P9** is earned for the correct answer 15.465041 rounded or truncated to three decimal places. A student who imports an answer from A and shows the imported answer times 15 earns **P8** and **P9**.

<u>Observations and recommendations for teachers:</u>

(1) C was given to identify the curve defined by $r(\theta) = 2\sin^2\theta$. Those students that use C in their calculations did not earn points. For example, in part b, $\int_{\pi/6}^{5\pi/6} \left(C^2 - \left(\frac{1}{2}\right)^2 \right) d\theta$ did not earn **P3**. In general, students should write down and define the functions they will use in the problem. It is helpful to use those names throughout the problem.

(2) Many of those students who did not earn the entry point for part a found $\frac{dy}{dx}$ or calculated an average rate of change. A similar issue with understanding extends to part d. Some of those students who missed part d found $\frac{dy}{dx}$ or used an arc length calculation. Better understanding of the meaning of $\frac{dy}{dx}, \frac{dx}{d\theta}, \frac{dy}{d\theta}, \frac{dr}{d\theta}$ would help students work with polar problems at the calculus level.

(3) For part b, many wrong answers were presented for finding the intersection points between the two polar curves. Additionally, errors with the integrand included missing squares $\left(r(\theta) - \frac{1}{2} \right)$ or incorrectly squaring the difference $\left(r(\theta) - \frac{1}{2} \right)^2$. The use of C was not accepted for $r(\theta)$.

(4) Students were given $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ in part c. A number of students with wrong answers tried using $\frac{dy}{dx}$. Some students presented vertical tangent arguments without success. For the candidates test, students need to evaluate all the candidates. Inequality statements in the candidates chart are not sufficient justifications.

(5) Not all students recognized the difference between a global and local argument. Teachers should consider emphasizing that the first and second derivative tests are local arguments and establish the criteria necessary to present a global argument.

(6) It is important for teachers to hold students to high standards for communication and notation. It may be misleading when P1 and P4 ignore linkage errors and other miscommunication errors. Teachers should require correct linkages and communication despite some relaxing of standards in scoring the exam as of 2025.