Problem Overview:

A student begins reading at time $t = 0$ and continues reading for the next 10 minutes at a rate modeled by the differentiable function $R(t)$, where $R(t)$ is measured in words per minute. Certain values of $R(t)$ are shown in the table below.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

Part a:

Students had to approximate the value of $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Work leading to the answer and the units of measure had to be shown.

Part b:

Students were asked if there is a value c with $0 < c < 10$ such that $R(c) = 155$ and to justify the answer.

Part c:

Students had to approximate the value of $\int_0^{10} R(t)\,dt$ with a trapezoidal sum and the three subintervals indicated by the data in the table.

Part d:

A teacher also begins reading at time $t = 0$, continuing to read for the next 10 minutes. The teacher's reading rate in words per minute is given by $W(t) = -\frac{3}{10}t^2 + 8t + 100$. Students had to calculate how many total words the teacher reads in these 10 minutes, showing work that leads to the answer.

Comments on student responses and scoring guidelines:

Part a worth 2 points

P1 was earned for the answer accompanied by supporting work showing a difference quotient and using values from the table. Expressions like $\frac{100-90}{2-0}$ or $\frac{100-90}{2}$ or $\frac{10}{2-0}$ earned **P1**. The expressions $\frac{10}{2}$ and $\frac{R(2)-R(0)}{2-0}$ did not earn **P1**, the first showing no difference(s) and the second not showing values pulled from the table. **P2** was earned for correct units with or without a numerical answer. Acceptable units are such as "words per minute per minute," or "words per minute squared," or "w/m^2." Many responses incorrectly stated "words per minute." Overall, responses were somewhat more successful computing the average rate of change than in providing correct units. Some responses found the average rate of change but then tried to apply this to an equation of a tangent line.

Part b: worth 2 points

Since the Intermediate Value Theorem will be applied, continuity must be established. This had to be done by noting that the given differentiability of $R(t)$ implies continuity and doing this earned **P3**. In order to earn **P4**, a response had to indicate the correct answer as well as the inequalities $R(0)$ (or $R(2)$ or $R(8)$) <155 and $R(10)>155$ or more directly, by using an extended inequality as in $R(0)<155<R(10)$. **P4** was not earned unless the continuity of $R(t)$ was asserted, whether or not continuity had been justified previously, and **P3** had been earned.

Part c: worth 2 points

Showing a form of a trapezoidal sum earned **P5.** This needed to include the sum of three terms, each a product of two factors, with one of the two factors incorporating $\frac{1}{2}$ as part of the product. This work may contain at most one error. For example, $\frac{90+100}{2}(2)+\frac{1001=150}{2}(2)+\frac{150+162}{2}(2)$ earned **P5**, but there is one error since the second factor of 2 should have been a 6. In the case of only one error **P5** was earned, but the response would not be eligible for **P6**, the answer along with supporting work.

$\frac{R(0)+R(2)}{2}(2)+\frac{R(2)+R(8)}{2}(6)+\frac{R(8)+R(10)}{2}(2)$ earned **P5**, but not **P6** until values from the table were used. Most students were using values from the table. The response $\frac{90+100}{2}(2)+\frac{100+150}{2}(6)+\frac{150+162}{2}(2)$ earned both **P5** and **P6**. **P6** was earned with that last response even if subsequent attempts at simplification contained errors.

Alternatively, students could have presented the average of correct left and right Riemann sums.

Part d: worth 3 points

P7 was earned for using $W(t)$ as the integrand of either a definite or indefinite integral, with or without the differential *dt*. **P8** was earned for the antiderivative of $W(t)$, with or without the constant of integration. Some students, unfortunately, had arithmetic difficulties with the coefficients in the antiderivative.

P9 was earned for the answer, simplified or not. Showing $-\frac{1}{10}\cdot 1000+4\cdot 100+100\cdot 10-\left(-\frac{1}{10}\cdot 0+4\cdot 0+100\cdot 0\right)$ or equivalent earned **P9,** despite any errors in subsequent attempts at simplification. **P8** must have been earned in order for a response to be eligible for **P9**.

<u>Observations and recommendations for teachers:</u>

(1) Most students seemed to know how to approximate $R'(1)$ using the average rate of change of R. Students need to be reminded that showing work on the exam is required. Regardless of how simple the arithmetic is in computing a difference quotient, showing the work for **P1** means showing that differences and a quotient are being used.

(2) That so many students had difficulty with **P2**, the units point, is indicative, in this reader's experience, of general such difficulty with units requiring a squared denominator. In part a of this question, units should be words per minute^2. The meaning of this is best (and should be) taught by noting that "words per minute" are changing. In fact, the value of $R'(1)$ is the rate of change in the number of words being read per minute. That is the rate of increase in the rate. R' is the rate of change of the rate given by R. $R'(1)$ found from the difference quotient is an estimate of how much that rate R (measured in words per minute) changes during the one minute (per minute) after $t=1$. We finally arrive at "words per minute per minute," because "per minute" is used twice once the second rate is applied to the first.

Teachers should deal with time as a unit being used in a variety of contexts, including velocity and acceleration. (Note: Do NOT let students refer to all second derivatives in such a context as "acceleration.") For other examples, see AP Calculus Exam questions 2016AB/BC1 part a and 2017AB2 part a.

(3) There are three intervals indicated by values of R in the table showing 155 as between the endpoints, using $R(10)$ as the right-hand endpoint. To establish the existence of *c* with $R(c)=155$ requires that R be continuous. Use of the Mean Value Theorem requires differentiability which we have in the given information. However, a difference quotient using the intervals mentioned above will not yield 155. This leaves us with having to apply the Intermediate Value Theorem. Unless observation of the question leads one immediately to use of the IVT, students should be prepared to consider use of both theorems as possible ways to approach this problem. Another reason to consider both possible theorems is to be certain that if a theorem is named, it is the correct one being used.

(4) Referring again to part b, the continuity of R must be mentioned in order to show that correct reasoning is being applied, along with the specific observation (an inequality) of the number 155 being a possible value $R(c)$ for $0 < c < 10$. To complete the argument, the continuity must be justified by the important fact that the differentiability of R *implies* continuity. This is so important that teachers should not only emphasize it because of how it is required on the exam; but also, teachers should discuss the proof of this fact in class.

(5) The trapezoidal method is often required on the AP Calculus Exam. Some texts and materials show a "formula" for the trapezoidal rule which can tend to mislead students into the use of intervals of equal length. Teachers should show application of the trapezoidal rule simply by using areas of trapezoids. Note that the intervals in the table of values in this question are not all of the same length. That is common on the AP exam.

(6) Part d requests information about a total amount or quantity given the rate $W(t)$ at which that is changing over a time interval. That can always be found as the definite integral of the rate of change, here $\int_{t_1}^{t_2} |W(t)|\,dt$. In the case where the rate is always positive, this is equivalent to $\int_{t_1}^{t_2} W(t)\,dt$. Student work on part d indicated that more practice is needed showing work by hand in computing antiderivatives as straightforward as an antiderivative of a polynomial with integer coefficients.