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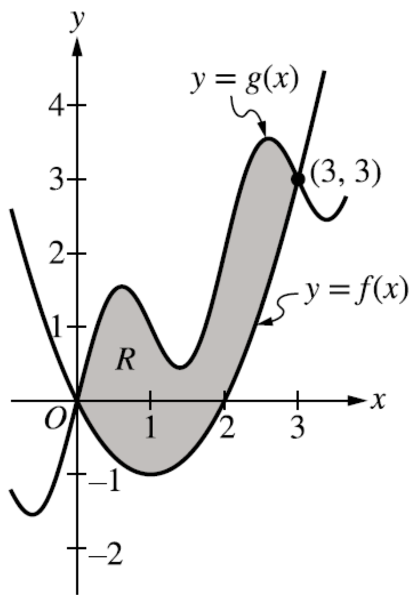
**Problem Overview:**

Students were given the graph below showing a region $R$ bounded by the graphs of $f(x) = x^2 - 2x$ and $g(x) = x + \sin(\pi x)$. Students were reminded that calculators should be in radian mode.




**Part a:**

Showing the setup for calculations, students were to find the area of $R$.

**Part b:**

Showing the setup for calculations, students were to find the volume of a solid with base $R$. At each $x$, a cross section perpendicular to the $x$-axis is a rectangle with height $x$ and base in region $R$.

**Part c:**

A solid is generated when $R$ is rotated about the line $y = -2$. Students had to write, but not evaluate, an integral expression for the volume of this solid.

**Part d:**

Students were given that $g'(x) = 1 + \pi\cos(\pi x)$. Students had to find the value of $x$ at which the line tangent to the graph of $f$ is parallel to the line tangent to the graph of $g$ for $0 < x < 1$.

**Comments on student responses and scoring guidelines:**

**Part a** worth 2 points

A definite integral was needed in order to compute the area between the two given curves. **P1** was earned for an integrand of form $f-g$, $g-f$, $|f-g|$ or $|g-f|$ with or without the differential *dx*. This could also be presented as the difference of two definite integrals with integrands $g(x)$ and $f(x)$. **P2** was earned for the correct answer with or without supporting work. Miscommunication or linkage errors were not penalized. For example, $\int_0^3 (f-g)\,dx = 5.137$ earned both points **P1** and **P2** despite the linkage error. The exact answer $\frac{9\pi+4}{2\pi}$ was occasionally presented.

**Part b:** worth 2 points

In order to earn **P3**, students were expected to present the area of a cross section in a definite integral as $x(f-g)$ or $x(g-f)$. There needed to be exactly two non-constant factors, at least one of which was one of the expected factors. **P4** was earned for the correct answer, with or without supporting work. As in part a, linkage or miscommunication errors did not impact scoring. The exact answer $\frac{27\pi+12}{4\pi}$ was occasionally presented.

A common misconception was the presentation of $(f(x)-g(x))^2$ as the integrand. This earned **P3**. Some responses included $x$ as a third factor along with $(f(x)-g(x))^2$, and this being more than two factors did not earn **P3**.

**Part c:** worth 3 points

**P5** was earned for a form $R^2-r^2$ of an integrand that had to be in a definite integral where at least one of $\{R,r\}$ is correct or a difference between *g* and a nonzero constant, and the other is correct or a difference between *f* and a nonzero constant. **P6** was earned for correct expressions for $R$ and $r$ written as an integrand in a definite integral in the form $R^2-r^2$. The form $r^2-R^2$, as above, was also acceptable for the earning of **P6**.

**P7** was earned for the entire correct definite integral including the constant $\pi$, the correct limits, and *dx*. If $R^2-r^2$, was reversed, the constant had to be $-\pi$ in order to earn **P7**.

**Part d:** worth 2 points

**P8** was earned for demonstrating that $f'$ had to be the same as $g'$ (where the slopes of lines tangent to the graphs of $g$ and $f$ are equal). This is best expressed as $f'=g'$ but could be written using the expressions for $f'$ and $g'$. **P8** could also be earned if the relationship between the slopes of these tangent lines was expressed verbally. **P9** was earned for the correct answer with or without supporting work. Most students entered part d knowing that parallel tangent lines have the same slope.

**Observations and recommendations for teachers:**

(1) In all four parts of this question, the functions in integrands or use of their derivatives must be written to demonstrate conceptual understanding. The functions $x^2 - 2x$ and $x + \sin(\pi x)$ are $f$ and $g$ respectively. The names of these functions should be used rather than the explicit expressions in order to avoid possible copy errors, miscommunication, or poor calculations in subsequent work.

(2) The area of a region bounded by two curves is a basic application of a definite integral. Although the antiderivatives in part a of this question could be computed by hand, there is no reason to do so. This is because use of a calculator was allowed. To do this by hand takes time and leaves room for errors in computation.

(3) In part a of this question, some students had misconceptions leading to poor calculation of the distance between the two curves, apparently because one of the graphs dipped below the $x$-axis. The vertical (or horizontal) distance between two points is merely the result of subtracting the lesser from the greater of the appropriate coordinates. No absolute value need be used. No special consideration need be given to the area lying below the $x$-axis.

(4) Volumes computed in AP Calculus typically involve a cross section area in a definite integral. This should be practiced with a variety of cross sections such as triangles, semicircles, rectangles and squares.

(5) In part c, students needed to know how to set up the area of the special cross section which forms a washer. Some students did not. Also, as in part a, some students showed errors in part c calculating the distance between the graphs of the functions and the axis of rotation, likely because this axis was below the $x$-axis. Again, these distances are simply the result of subtracting as in $f-(-2)$ and $g-(-2)$.

(6) To earn the last point in part c, some details needed to be correct in the definite integral including the differential $dx$. Teachers should always emphasize proper inclusion of the differential.

(7) For two lines tangent to graphs of functions to have the same slopes, the derivatives of those functions must be equal. Most responses in part d seemed to demonstrate awareness of this concept. To finish answering the part d question, students had to solve the resulting equation. Whether using a solver or another method, students should know how to use a calculator to get a decimal approximation of a solution to an equation.