2024 BC5 Marshall Ransom, Georgia Southern University

Problem Overview:

The function $f(x)$ is given as twice differentiable for all x with $f(0)=0$. Some values of $f'(x)$ are given in the table below.

x	0	π	2π
$f'(x)$	5	6	0

Part a:

Students were given that for $x \geq 0$ the function h is defined by $h(x)=\int_0^x \sqrt{1+\left(f'(t)\right)^2}\,dt$, and asked to find the value of $h'(\pi)$ showing work that leads to the answer.

Part b:

Students were asked what information $\int_0^{\pi} \sqrt{1+\left(f'(x)\right)^2}\,dx$ provides about the graph of f.

Part c:

Using Euler's method starting at $x=0$ with two steps of equal size, students had to estimate the value of $f(2\pi)$, showing computations that lead to the answer.

Part d:

Students were asked to find $\int (t+5)\cos\left(\frac{t}{4}\right)dt$, showing work leading to the answer.

Comments on student responses and scoring guidelines:

Part a worth 2 points

The first point is for using the Fundamental Theorem to arrive at $h'(x)=\sqrt{1+\left(f'(x)\right)^2}$. By itself, $\sqrt{1+\left(f'(\pi)\right)^2}$ also earned the first point. The second point was for the answer, and $\sqrt{1+6^2}$ earned both points. A response showing a linkage error such as $h'(x)=\sqrt{1+\left(f'(x)\right)^2}=\sqrt{37}$ earned at most 1 of the 2 points. Another error seen was $h'(x)=\frac{d}{dx}\int_0^{\pi}\sqrt{1+\left(f'(x)\right)^2}\,dx$. These types of errors were not uncommon.

Part b worth 2 points

The first point was earned for using "length" or "arc length." If the response referred to "distance" or "distance traveled" it had to explicitly connect to the graph of f. The second point was earned for connecting the interval $[0, \pi]$ to "arc," "arc length" or "distance." For example, the response "distance traveled on $[0, \pi]$" earned the second point but not the first; however, "arc length on $[0, \pi]$" earned both points.

Part c worth 2 points

The first point was earned for two steps of Euler's method, using the correct expression for $\frac{dy}{dx}$, with at most one error. The second point was earned for the answer 11π. In the case of one error in Euler's method, the second point was not earned. About half of the responses showed the linear approximation approach to the computation, and half showed the tabular approach. Using the tabular approach required that the answer be clearly indicated. This approach did not have to be labeled if completely correct. Using a third step in Euler's method resulted in a response not earning the answer point. At times it could be difficult for a reader to discern whether or not a third step was being used, since $f'(2\pi) = 0$, use of which also resulted in an answer of 11π.

Part d: worth 3 points

This part required showing work using integration by parts to compute an indefinite integral. The first point was earned for work showing correct *u* and *dv*, either for $\int (t+5)\cos\left(\frac{t}{4}\right)dt$ or $\int t\cos\left(\frac{t}{4}\right)dt$. (Some responses worked with the original integral split into the two parts $\int t\cos\left(\frac{t}{4}\right)dt$ and $\int 5\cos\left(\frac{t}{4}\right)dt$.) The second point was earned for applying $uv - \int v\,du$ to either $\int (t+5)\cos\left(\frac{t}{4}\right)dt$ or $\int t\cos\left(\frac{t}{4}\right)dt$. The third point was for the answer, including a constant of integration.

The tabular method could be used, the first point being earned for columns beginning with either $t+5$ and $\cos\left(\frac{t}{4}\right)$ or t and $\cos\left(\frac{t}{4}\right)$. The second point was earned for the expression $4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)dt$ if $t+5$ was used in the table or $4t\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)dt$ if t was used in the table. Once again, the third point was earned for the correct answer with a constant of integration.

<u>Observations and recommendations for teachers:</u>

(1) Students should practice using the Fundamental Theorem to compute the derivative of an integral expression, in particular with function limits. This is almost always on the AP Calculus Exam.

(2) Students continue to show linkage errors, equating a variable expression to a constant. Readers see this often. Teachers should be very strict about this type of error when grading student classwork and tests. Both

incorrect and correct approaches should be shown to students in various contexts. This is a far too common error on the AP Calculus Exam FRQs. The following shows the beginning of correct work for part a ending in a linkage error: $h'(x) = \sqrt{1+\left(f'(x)\right)^2} = \sqrt{1+6^2}$. Sometimes, students continually use the equal sign in writing work horizontally. Starting new work on the next line, vertically showing work, can be a better approach.

(3) BC students should be very familiar with formulas for arc length, both for single variable functions and for parametric expressions. It's a good idea for teachers to ask of students that they describe their results verbally.

(4) Readers could not infer, even from a perfectly organized table, which entry was the answer for the Euler method approximation. Students had to make clear the final result. Entries in the table should be clearly labeled.

(5) The standard approach to integration by parts is to begin with the parts labeled as *u* and *dv*. Doing so in a manner that would lead to an answer typically earns the first point on the AP Calculus Exam. The second point is earned for using $uv - \int v\,du$. The answer point is for correctly combining the parts, including the result for $\int v\,du$ and a constant of integration.

(6) Unfortunately, also in part c, there were far too many examples of mishandling the constant when integrating $\sin\left(\frac{t}{4}\right)$. This can be practiced when first learning about *u*-substitution by starting with the substitution $u = \frac{t}{4}$. It should be practiced often enough so that for most students the result can be obtained quickly when *u* is linear. There are many other examples that involve a function *u* that is linear such as $\int (3x+4)^8\,dx = \frac{1}{3}\bullet\frac{(3x+4)^9}{9} + C$ and $\int \frac{dx}{(1-5x)} = -\frac{1}{5}\ln|1-5x| + C$.

(7) Students should be reminded that a constant of integration must be included in the result for an indefinite integral.