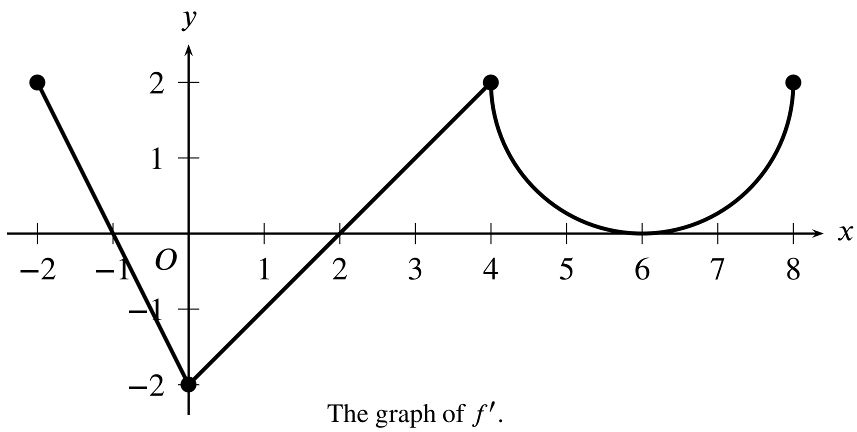
Problem Overview

The student was given a graph of the derivative of a function f on the interval $[-2, 8]$. The student was also given that $f(2) = 1$. The graph of f' is shown below, and consists of two line segments and a semicircle.



The graph of f'.

Part a

Students were asked to determine whether f has a relative minimum, maximum, or neither at the point where $x = 6$, and to give a reason.

Part b

Students were asked to determine the intervals where f is concave down, and to give a reason.

Part c

Students were asked to evaluate $\lim\limits_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, and to justify the answer.

Part d

Students were asked to find the absolute minimum value of f on $[-2, 8]$, and to justify the answer.

Comments on Student Responses and Scoring Guidelines

Part a

This part of the question could earn the student only a single point. To earn this point, the student had to indicate that there was neither a maximum nor a minimum at $x = 6$ because f' did not change sign at $x = 6$. This could be indicated in multiple ways: the student could say that f' did not change from positive to negative or from negative to positive, or that f' is positive on the intervals $(4, 6)$ and $(6, 8)$. Since the graph given was f', the student could write "the graph" or "the derivative" instead of f'. However, if the student referenced "it" (as in "it doesn't change sign") or referenced "the slope" or referenced the behavior of f, the student did not earn the point.

Part b

This part of the question could earn the student two points. The first point is earned by reporting the correct intervals of $(-2, 0)$ and $(4, 6)$. No other response earned this point. However, if the student presented one of the two intervals – either $(-2, 0)$ or $(4, 6)$ – and *no other incorrect intervals*, the student did not earn this point but remained eligible to earn the second point.

The second point was earned provided the student listed both correct intervals, or exactly one of the correct intervals and no other incorrect intervals. To earn this second point, the student must have referenced the behavior of f', not f'' (which was not given). The expected response that earned the point indicated that f' is decreasing on these intervals. Since the students were given the graph of f', the student could write "the slope of the graph" or "the slope of f'" but could not reference f'' without connecting f'' to the graph given. If the student referenced f'' without this connection, the student did not earn the point.

If the student listed any incorrect intervals (even in the presence of one or both correct intervals), neither of the two points were earned.

Part c

Three points are available to the student in part (c). To earn the first point, the student had to show that the limits of the numerator and denominator are zero separately:

$$\lim_{x\to 2}(6f(x) - 3x) = 0 \quad \text{and} \quad \lim_{x\to 2}(x^2 - 5x + 6) = 0.$$

This could be indicated with arrows ("$\longrightarrow 0$") as long as the numerator was separated from the denominator.

The second point was earned if the student correctly exhibited use of L'Hôpital's rule by writing a limit of the ratio of the derivatives of the numerator and denominator. Readers needed to see the limit attached to the ratio of $6f'(x) - 3$ and $2x - 5$ as in

$$\lim_{x\to 2}\frac{6f'(x) - 3}{2x - 5}.$$

At least one of the derivatives in this ratio must be correct to earn the point. The ratio of derivatives without a limit attached, or two incorrect derivatives, did not earn the second point.

The third point was earned if the student computed the correct answer, 3, with supporting work. The supporting work could include work that did not earn either of the first two points, such as a ratio of correct derivatives with no limit attached.

Part d

In the last part of this problem, three points were available: one point for showing evidence of considering $f'(x) = 0$, one point for the the so-called "candidate's test", and one point for the answer.

Earning the first point was easy. The student simply had to indicate that they considered $f'(x) = 0$ by writing "$f'(x) = 0$", by writing "$f'(2) = 0$", by writing "the critical points are $x = -1, 2, 6$", or by saying that f' changed sign at $x = -1$ or at $x = 2$.

The second point was more difficult for students to earn. Students had to present an argument that ruled out $x = -1$ and $x = 6$ as possible locations of absolute minima, and an argument that explained why the absolute minimum was not at the endpoints $x = -2$ and $x = 8$. The easiest way to do this was to compute the values of f at each of these five points, and show that $f(2)$ is the smallest:

x	$f(x)$
-2	3
-1	4
2	1
6	$7 - \pi$
8	$11 - 2\pi$

However, another approach involved students arguing that other points were not minima. Students could argue that $x = -1$ gave a local maximum by writing $f' > 0$ for $x < -1$ and $f' < 0$ for $-1 < x < 2$. Students could rule out $x = 6$ as an extrema by referencing the work in part (a). Students could also show that $f(-2) > f(2)$ and $f(8) > f(2)$ using an area or derivative argument.

If the student did not rule out $x = -1$ as giving a possible minimum, the student did not earn this point and was not eligible for the answer point. If the student did not rule out $x = 6$, the student did not earn this point but remained eligible for the answer point. If the student used a table of function values (as in the candidate's test), then one incorrect value did not earn the student this point, but they remained eligible for the answer point. More than one mistake in computing function values did not earn the student this point or the answer point.

The last point was for stating the correct minimum value of f, which was 1. The student must have earned at least one of the first two points to be eligible to earn this point (except in the case of not ruling out $x = -1$ or two or more incorrect function values). However, the student had to explicitly state the value 1. The minimum value could not be contained in an ordered pair or presented as $f(2)$ since the problem explicitly asked for a value.

Observations and Recommendations for Teachers

(1) Students should read the problem. Many students interpreted the given graph of f' as a graph of f. This rendered any work they did incorrect and they did not earn any points.

(2) Students should justify answers based only on the information given. Students went from the graph of f' into immediately talking about f or f'' without connecting f or f'' back to the given information. The student who did not make the connection to the given information did not earn points.

(3) Students should stop writing. Many students talked themselves out of points by writing a long paragraph which usually contained correct answers worth some points, but the student continued writing and in doing so, wrote things that were contradictory, incorrect, or false. This resulted in penalizing the student by taking away some of their earned points. A good rule of thumb to tell students is this: *If you are writing more than a sentence, you are doing it wrong!*

(4) Here are some some things students should avoid writing in problems like part (a). Many students wrote something like the following.

"$x = 6$ is neither because f' is positive for $x < 6$ and $x > 6$."

"$x = 6$ is neither because f' is positive on both sides of $x = 6$."

"$x = 6$ is neither because f' is positive to the left and the right of $x = 6$."

"$x = 6$ is neither because f' is positive before and after $x = 6$."

What about the interval $[-1, 2]$ *where* f' *is negative?* These student responses are false statements and none of these earned the point. Another false statement by students that did not earn the point was

"$x = 6$ is neither because f' is positive on the interval $(4, 8)$."

Unfortunately, $f'(6)$ is not positive; $f'(6)$ is zero. A few students said that f' was nonnegative on $(4, 8)$ and that earned the point. However, these responses did not earn the point:

"$x = 6$ is neither because f is increasing on the interval $(4, 8)$."

"$x = 6$ is neither because f has an inflection point at $x = 6$."

How does the student know these? They know it because f' is positive on $(4, 6)$ and on $(6, 8)$, and $f'(6) = 0$ – which is what they should have said, and saying that earned the point! The student should connect f to f' and in doing so, talk about f', not f. Connecting this is made worse by ambiguity. Many responses in part (a) were

"Neither at $x = 6$ because it does not change sign."

The word "it" is too ambiguous and this did not earn the point.

(5) Here are some things students should avoid writing in problems like part (b). Many students wrote something like the following in justifying the intervals of concavity.

"This is where the slope is decreasing."

What is meant by "slope"? Since the graph is f', is the student talking about the graph which is the "slope" of f, or is the student talking about the slope of the given graph? This is too ambiguous and did not earn the point. Here's another example.

"This is where f'' is negative."

While true, the student did not connect this back to the given graph of f' and so this did not earn the point. And speaking of connecting back to the given information, student responses such as the following only earned the first of the two points available in part (b).

"f is concave up on $(-2, 0)$ and $(4, 6)$ because
if f' is decreasing on an interval, then f is concave up there."

This only earned the point for the correct intervals and not the point for the justification because the student did not explicitly connect the stated property to the given information. The presence in student work of an *if-then* statement is read as the statement of a theorem (or as a recipe for doing the problem but *not* actually answering the question) unless the student connects the statement to the given information. Consider the following response and compare it to the previous one.

"If f' is decreasing on an interval, then f is concave up on that interval.
f' is decreasing on $(-2, 0)$ and $(4, 6)$ so f is concave up there."

This response earns both points in part (b). Indeed the first sentence is entirely unnecessary to earn both points. At no time are students required or asked to write *if-then* statements.

(6) Students should be familiar with and use mathematical terminology. In part (b), some students claimed that f is never concave down on any open interval because f has no open intervals. I can only assume that since f was defined on $[-2, 8]$, these students thought that since f is only defined on a closed interval, one cannot use open intervals to describe portions of f.

(7) In part (c), the student had to carefully handle the the limits of the numerator and denominator. These needed to be shown as separate limits, with limit notation. That is, writing

$$\lim_{x\to 2}\frac{6f(x)-3x}{x^2-5x+6}\begin{array}{l}\nearrow 0\\[1ex] \searrow 0\end{array}$$

did not earn the point. Likewise, writing

$$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \frac{0}{0}$$

or linking $0/0$ to any simplified or evaluated version of this limit with an equals sign did not earn this point. Some students linked the given limit with "$= 0/0$" and then presented separate limits. Those students earned the first point but did not earn the answer point, being penalized for bad communication.

(8) Students should stop doing arithmetic. Many correct limits of the ratio of derivatives in part (c) did not earn the answer point because of bad arithmetic. This Reader saw many examples of students simplifying $4-5$ to 1 rather than -1. Students should be repeatedly instructed that unsimplified correct answers earn the point.

(9) Students should stop doing unecessary algebra. Many students noticed that the denominator of the limit in part (c) was factorable. Students then attempted to factor the numerator to hopefully cancel out a factor. Of course, this did not lead to the correct answer and resulted in many students simply concluding that the limit does not exist. However, some students attempted to be clever: they used the fact that $f(2) = 1$ to make the numerator $6f(2)-3x = 6-3x = -3(x-2)$. In doing this the factors of $x-2$ cancelled, leaving

$$\lim_{x\to 2} \frac{-3}{x-3}.$$

Students then substituted $x = 2$ in order to evaluate the limit and obtained the "correct" answer, 3. Of course, this method is incorrect and earned no points.
On the other hand, a few students realized that the graph of f' is $f'(x) = x-2$ on the interval $[0, 4]$ and so used the *antiderivative* of $f'(x) = x-2$ in the limit. They then obtained the correct answer without having to resort to L'Hôpital. The antiderivative is $\frac{1}{2}x^2 - 2x + C$, and using $f(2) = 1$ gives $f(x) = \frac{1}{2}x^2 - 2x + 3$ on $[0, 4]$. Then

$$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{3x^2-15x+18}{x^2-5x+6} = \lim_{x\to 2} \frac{3(x^2-5x+6)}{x^2-5x+6} = 3.$$

While this solution was not common (perhaps 5 responses out of 350,000), it did earn all 3 points. Correct mathematics leading to the correct answer is always acceptable on the AP Exam.

(10) In part (c), students presented bad derivatives of $6f(x)-3x$. We saw students dropping the coefficient 6 and apparently believing that the derivative of $3x$ is 0. More egregious were student errors in writing the derivative of $6f(x)$ as 6, or even 0. Luckily, to earn the second point in part (c), only one correct derivative was required, but such mistakes rendered the answer point unearned. Students should have practice finding the derivatives of functions in function notation.

(11) Students should avoid writing too much. Students used words like "velocity" and "acceleration" in place of f' and f''. Any justifications using these words were not accepted as students were not given a position or velocity function. Students tried to justify that $f(2) = 1$ is the absolute minimum using many different kinds of verbal arguments, but any kind of verbal argument had to be stated very carefully. This Reader did not see any such verbal argument in part (d) earn more than the first point. Students need to justify answers using mathematical reasoning.