2023 BC6 Marshall Ransom, Georgia Southern University

Problem Overview:

It is given that the function f has derivatives of all orders for all real numbers. It is also known that $f(0)=2$, $f'(0)=3$, $f''(x)=-f\left(x^2\right)$, and $f'''(x)=-2x \bullet f'\left(x^2\right)$.

Part a:

Students were asked to find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Also, the fourth-degree Taylor polynomial for f about $x=0$ was requested, along with work leading to that answer.

Part b:

Students were told that the fourth-degree Taylor polynomial for f is used to approximate $f(0.1)$. Students then had to use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$.

Part c:

The following information about the function g was given: $g(0)=4$ and $g'(x)=e^x f(x)$. Students were asked to write the second-degree Taylor polynomial for the function g about $x=0$.

Comments on student responses and scoring guidelines:

Part a worth 4 points

The first point was earned for evidence of the product rule in computing $f^{(4)}(x)$. A typical error was in the presentation of $f^{(4)}(x)=-2f'\left(x^2\right)+(-2x)f''\left(x^2\right)$ which omits the chain rule. This still earned the first point. The second point was for a completely correct expression of $f^{(4)}(x)$. Thus the chain rule error earning the first point because of use of the product rule would not have earned the second point for $f^{(4)}(x)$. The third point was for showing two correct terms of the requested Taylor polynomial, and the fourth point was for showing all of the requested terms. A response earning the first point but not the second could use a consistent value for $f^{(4)}(x)$ in computing the fourth degree term of the Taylor polynomial. A response could not earn the fourth point if the third degree term was not 0, if the given terms (even if correct) concluded with “+…”, or additional terms of higher degree were presented.

Some students, doing correct work, presented a linkage error which would not earn the second point. We note such an error as in $f^{(4)}(x) = f^{(4)}(0)$ rather than $f^{(4)}(x)\big|_{x=0} = f^{(4)}(0)$.

Part b: worth 2 points

The first point was for a form of the remainder such as $\frac{15}{5!}(0.1)^5$ or more formally, $\frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!}(0.1)^5$.

In order to earn the second point, a response had to include an inequality such as $\frac{15}{5!}(0.1)^5 \le \frac{1}{10^5}$. Since the LaGrange error bound is an upper bound on the error of the approximation, an equation would not earn the second point as in $\text{Error} = \frac{15}{5!}(0.1)^5 = \frac{1}{10^5}$ or $\text{Error} = \frac{1}{10^5}$.

Part c: worth 3 points

In order to show the second-degree Taylor polynomial for the function g, $\frac{d}{dx}g'(x) = e^x f'(x) + e^x f(x)$ had to be computed. This earned the first point. Since this work was for forming the second-degree Taylor polynomial about $x = 0$, either $e^0 f'(0) + e^0 f(0)$ and $f'(0) + f(0)$ would earn the first point. The second point was earned for the first two terms, and the third point was for all three correct terms. A polynomial of the form $4 + 2x + ax^2$ earned the second point, even without supporting work for the first two terms. Work supporting the value of a was required for the third point to be earned.

A few students used an alternate solution. This started with multiplying the known MacLaurin series for e^x by terms found for the Taylor polynomial in part a. Reporting $\left(1 + x + \frac{x^2}{2} + \cdots\right)\left(2 + 3x - x^2 + \cdots\right) = 2 + 5x + \cdots$ earned the first point under the alternate scoring guidelines. The second point was for two correct terms and the third point was earned for all three terms. Work required computing $\int (2 + 5x + \cdots)dx = C + 2x + 5\frac{x^2}{2} + \cdots$ Because $g(0) = 4$, we see that $C = 4$, and the second-degree polynomial could be presented.

Observations and recommendations for teachers:

(1) Errors in using the product and chain rules were all too common on this BC question. This points to the need for reinforcement of these computational skills. One possible way to do this is to include a few basic review questions to start every test or quiz.

(2) Many students showed knowledge of how to set up a Taylor series. But some such students did not read the question carefully enough. Asking for a Taylor polynomial of a certain degree is not the same as asking for a Taylor series. Students who included in their responses "+…", or additional terms of higher degree than that which was requested lost a point despite knowing much about Taylor series. Students, especially sometimes good students, should always look once again at the wording of the question to see if their good work actually resulted in an answer to the specific question.

(3) The work for finding a LaGrange error bound on an approximation given by an nth-degree Taylor polynomial involves finding a maximum value for $f^{(n+1)}(x)$. It is not always easy to find that bound, but in this question the bound was given as 15. Then, multiply by $(x - center)^{n+1} = (0.1-0)^{n+1}$ and divide by $(n+1)!$. The result in this question, $\frac{15}{5!}(0.1)^5$, is a bound as indicated in the question by the language, "show that this approximation is within $\frac{1}{10^5}$." All too many students did not use this reinforcing language to realize that an inequality is required and lost a point on part b.

(4) As in part a, the product rule had to be used in part c. This resulted in some students losing points in both parts. But this error was deemed too significant to be forgiven the second time it appeared. This basic computational skill needs to be as automatic as application of the power rule. An error can cost students multiple points on the exam.