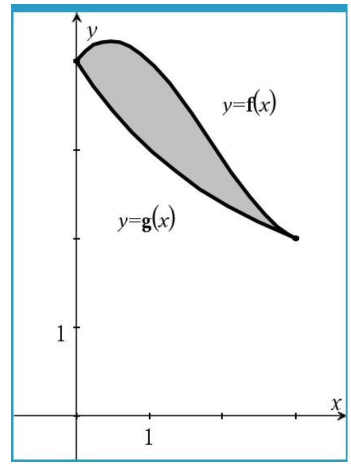
2022 BC-5 Dennis Wilson, Landmark Christian School

Problem Overview

The students were given the graph of two functions f and g on the interval $0 \le x \le 3$ as shown in the figure to the right. The region bounded by these two functions was shaded. The students were given an explicit expression for g which was $g(x) = \frac{12}{3+x}$ for $x \ge 0$. The students were not given an explicit algebraic equation for f but they were told that f was twice-differentiable, that $f(3) = 2$, and that $\int_0^3 f(x)\,dx = 10$. See note in "Observations and Recommendations for Teachers" for a possible explicit polynomial which meets the conditions given for f.



Part a

Students were asked to find the area of the region that was bounded by the graphs of f and g.

Part b

Students were asked to evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$ or to show that the integral diverges.

Part c

Students were told that a function h was defined by the equation $h(x) = x \cdot f'(x)$. The students were then asked to find the value of the integral $\int_0^3 h(x)\,dx$

Comments on Student Responses and Scoring Guidelines

Part a: worth 3 points

To complete Part (a), students needed to evaluate the definite integral of the difference of f and g on the interval from 0 to 3, that is, $\int_0^3 (f(x) - g(x))\,dx$. The first of three points for this part was earned for the integrand of this integral. Various forms of the integrand could earn the first point, notably $g(x) - f(x)$. Students who wrote the integrand in this form and did not correct it by communicating a positive value for the area would miss the third point for this part as their answer. As no explicit expression was given for $f(x)$, students could earn the integrand point by writing $10 - \int_0^3 g(x)\,dx$.
Students earned the second point for this part with an antiderivative of $g(x)$. Any constant multiple of either $\ln(3 - x)$ or $\ln|x - 3|$ could earn the point.

The third and final point was for the answer. Students could not earn the third point unless they had earned the first two points, that is, the reader did not need to consider the student's answer if the student incorrectly wrote the integrand or incorrectly computed the antiderivative. Students who used u-substitution to compute the definite integral but failed to properly update their limits of integration did not earn the third point.

Part b: worth 3 points

The first point for this part was earned by expressing the improper integral as a limit. This limit notation needed to be used correctly throughout the problem. It could not disappear and reappear. Students could not substitute infinity into their expression or perform arithmetic with infinity. If so, they forfeited this first point.
The second point was earned for finding an antiderivative of $\frac{144}{(3+x)^2}$ in the form $-\frac{a}{3+x}$ where $a > 0$. If $a \neq 144$, then students would not earn the third point, which was for a value numerically equal to 48.
If students used u-substitution to determine the definite integral, they needed to be careful in their change of variable. Their was a danger equating two integral which were not equal. If a student did so, then they would lose the third point.

Part c: worth 3 points

The first two points of this part were for correct use of integration by parts. The first point was awarded for identifying u and dv (or v') correctly. The second point was awarded for the correct application of integration by parts using the stated u and dv, whether these values earned the first point or not. Both points could simultaneously be earned by simply stating

$$x \cdot f(x) - \int f(x)\, dx,$$

with or without limits of integration. Students could earn the first two points using a tabular method. The columns of the table did not need to be labeled.
The third point was earned for an answer numerically equivalent to -4. The student need not simplify an arithmetic expression to earn the point.

Observations and Recommendations for Teachers

(1) A polynomial function that meets the conditions given in the problem is $f(x) = \frac{2}{9}x^3 - \frac{11}{9}x^2 + x + 4$. This could be useful in preparing technology to demonstrate the problem to students.

(2) The antiderivatives in part (a) and part (b) could be evaluated in many ways. Some students used u-substitution and in some of those cases students missed points from linkage errors. Students should be presented with a way of evaluating integrals that involve the composition of functions where the inside function is linear: $\int f'(mx+b)\, dx = \frac{1}{m} f(mx+b) + C$. Integrals of this type occur

on the non-calculator Free Response Question section, and this technique provides students with a quick and reliable method for evaluating them.

(3) Students continue to use arithmetic with infinity to reason through a solution to improper integrals. While this can be a helpful method for students, they need to be warned of the danger of losing points for such work. Students do make successful use of such incorrect symbolic manipulation without losing points. These students perform their arithmetic to the side of the body of their work and then cross out the offending arithmetic when finished. Students should be informed that readers do not score work that has been crossed out and so it cannot be counted against the student.

(4) Students should be encouraged to stop once an arithmetic expression for an answer is achieved. Simplification is not necessary and only steals valuable time that can be effectively used elsewhere. Students should be shown arithmetic expressions that earn full credit for Free Response Questions. To help students learn to not simplify arithmetic expressions and yet sharpen their mental arithmetic skills, consider adding multiple choices for the simplified answer which students can choose for extra credit on free response assessments.