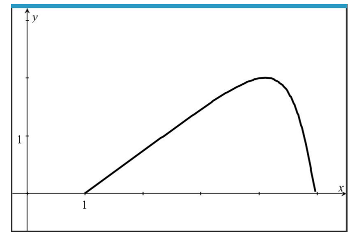
2022 BC-5 Dennis Wilson, Landmark Christian School

Problem Overview

The students were told that a particle moved along a curve such that its position at time t was given by $(x(t), y(t))$ for $0 \leq t \leq \pi$. The function $x(t)$ was not explicitly given and $y(t) = 2\sin(t)$. The path of the particle was given as a graph as shown in the figure to the right. Students were also told that $\frac{dx}{dt} = e^{\cos t}$ and that at time $t = 0$, the particle was at the position $(1, 0)$.



Part a

Students were asked to find the acceleration vector of the particle at time $t = 1$. Students were also reminded to show the setup for their calculations.

Part b

Students were asked to find the first time t at which the speed of the particle was 1.5 for $0 \leq t \leq \pi$. Students were once again reminded to show the work leading to their answer. A reminder of this type was given in all four parts of this problem.

Part c

Students were asked to find two values in this part. First, students were asked to find the slope of the line tangent to the path of the particle at time $t = 1$. Second, students were asked to find the x-coordinate of the position of the particle at time $t = 1$.

Part d

Students were asked to find the total distance traveled by the particle for the time interval $0 \leq t \leq \pi$.

Comments on Student Responses and Scoring Guidelines

Part a: worth 2 points

Each component of the acceleration earned one of the points with proper setup. To earned the point for $x''(1)$, students needed to show that it was equivalent to the derivative of $\frac{dx}{dt}$ at $t = 1$ along with the approximate numerical value of -1.444 or the exact value of $-e^{\cos 1}\sin 1$.
To earn the point for $y''(1)$, students needed to show that the second derivative of $y(t)$ with respect to t at $t = 1$ was equivalent to the approximate value of -1.683 (rounded) or -1.682 (truncated). Students could also give the exact value of $y''(1)$ as $-2\sin 1$.

A student could earn one of the two points if he or she stated the acceleration vector for time t as $\langle -e^{\cos t}\sin t, -2\sin t\rangle$ but failed to give the correct evaluation at $t = 1$.
Students who presented correct numerical values for both $x''(1)$ and $y''(1)$ without supporting work earned one of the two points.

Part b: worth 2 points

Students earned the first point for this part by equating a correct expression for the speed to the value of 1.5. Students were not required to explicitly state this in the form of an equation, nor were they required to enter into the problem by using the explicit functions for the components of the velocity, that is, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5$ was sufficient.
The second point was earned with the correct answer of $t = 1.254$. By itself, this numerical value could not earn either point. Students could earn the second point without earning the first point with a parenthesis error in the statement of the equation.
Many students failed to earn this point due to incorrectly rounding the time to 1.255. (See Observations (3)).
Others missed the second point by incorrectly giving the second time in the interval ($t = 2.358$) instead of the first. (See Observations (2)).

Part c: worth 3 points

The first point was earned for the slope of the line tangent to the path of the particle. A numerical value alone could not earn this point. Supporting work needed to show $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$. This could be done in various ways but the simplest was to express the exact value as $\frac{dy}{dx} = \frac{2\cos 1}{e^{\cos 1}}$.
The second point was earned for the definite integral $\int_0^1 e^{\cos t}\,dt$. The presence of the initial condition was not required for this point. Students could earn this point without expressing the differential in their integral. (See Observations (4)).
The third and final point was earned for the numerical value of $x(1)$.

Part d: worth 2 points

Students earned the first point by expressing a definite integral with a correct integrand. If the limits of the integral were incorrect, the second point was lost. Incorrect limits was a common mistake as many student used the interval $0 \le t \le 1$ from part (c) again in part (d).
Students who imported an incorrect expression for the speed from part (b) earned the first point but were not eligible for the second. An error from incorrect parenthesis was not deducted twice, so students making that error could earn both points on this problem.
The second point was earned for the correct numerical value of the distance traveled over the interval.

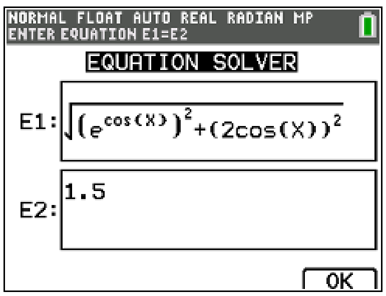
Observations and Recommendations for Teachers

(1) In years past, many student missed points for not showing the setup for their calculations. On this problem, students were reminded in each part to show this setup. This reminder served the students well. This problem had a higher percentage of 9's than any I have scored in my eight years as reader.

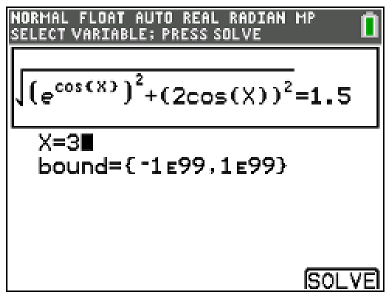
(2) In part (b) students were required to find the first of two solutions to the equation

$$\sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5.$$

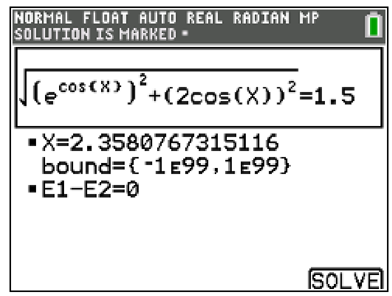
Most students found the first solution correctly, but others found the second solution and reported it as the answer. This could arise from use of the Numerical Solver on the TI-84. The x value used as a guess in the Numerical Solver is always the previous value calculated by the calculator. If students do not change the value, they could arrive at values they are not seeking.



In the Numerical Solver, student enter the left side of the equation as E1 and the right side as E2.

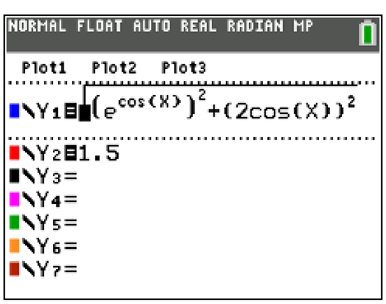


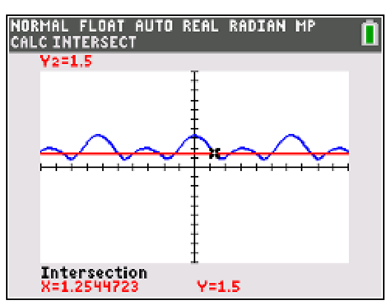
Students then have the ability to set the initial value of x used in searching for the solution as well as set the interval in which the search should be conducted.

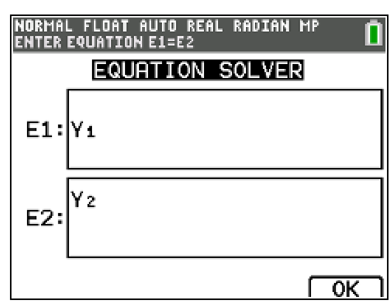


With an incorrect initial value of x, the second value is returned instead of the first.

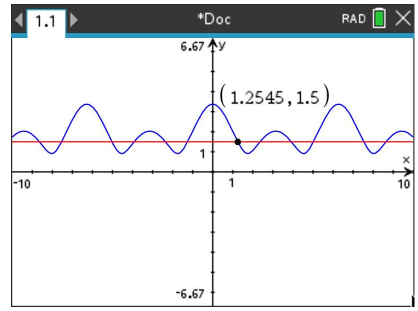
Students who exclusively use the Numerical Solver should be cautioned to always set the value of x themselves before solving. I recommend that students use the graphing window to solve the equation. While some skepticism is present about the tolerance used by the graphing window, the Numerical Solver could be used to verify the result later. Having $Y1$ and $Y2$ stored in the Numerical Solver makes this check happen very quickly as the intersection becomes the initial value.







(3) The approximate value for the time sought in part (b) was 1.2544723. Some students incorrectly reported the rounded value as 1.255. This could be due simply to students incorrectly rounding the fourth decimal place up and then rounded the third decimal place. It could also have arisen from students not displaying enough digits for the computation and the calculator rounded the fourth decimal place as seen in the screenshot from the TI-Nspire in the figure to the right. Students should be warned not to trust values represented with only four decimal places. I highly encourage my own students to adopt the practice of truncating answers for the AP exam to prevent such errors.



(4) As AP Calculus teachers, we hope that our students have learned the importance of writing the differential with every integral by the time they take the AP exam. Once sitting for this high stakes exam though, students can frequently lapse in performing this important task. In most places, this omission does not result in the student missing points. Questions which involve finding the value of a function given an initial value and its derivative (like part (c)) is where students often lose points for the missing differential. If students add the initial value after the an integral written with an omitted differential, the reader cannot decide if a student intended the initial value to be part of the integrand or not. And so, students lose the point. To combat this, students should be encouraged to write the sum in the opposite order, that is with the initial value first and the definite integral added to it. This is often expressed in textbooks as the Net Change Theorem.

$$f(b) = f(a) + \int_a^b f'(x)\, dx$$

(5) Students need to be admonished early (well before reviewing for the exam) that they will not need to analytically compute a derivative or antiderivative by hand on the calculator active Free Response Questions. Some students spent time erroneously searching for an antiderivative of the function $x'(t) = e^{\cos t}$. All the students who attempted this wasted time and arrived at incorrect answers. It is recommended that students be shown multiple examples of functions which do not have an antiderivative that can expressed without an integral.

(6) While this question concerns motion in two dimensions, AP Calculus AB teachers should consider making use of each function individually for problems concerning rectilinear motion.

(7) The graph given to students proved to be a red herring. There was no useful information that was gleamed from it by students who did well and it only served to confuse students who struggled.