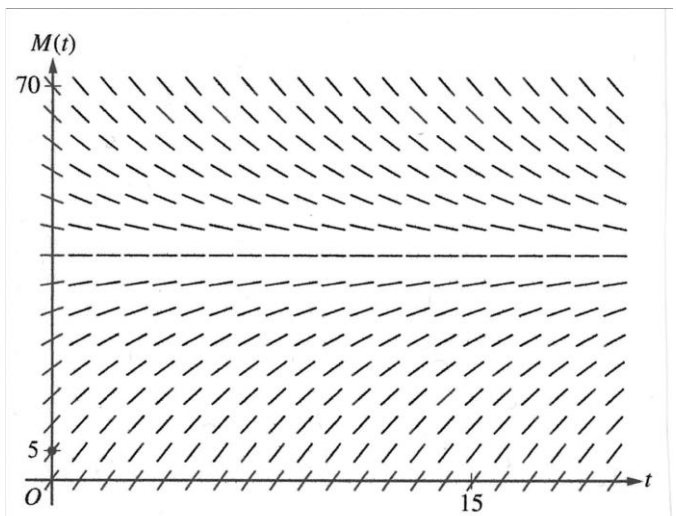
2023 AB/BC3 Marshall Ransom, Georgia Southern University

Problem Overview:

An increasing function M models the temperature of a bottle of milk, taken out of a refrigerator and placed in a pan of hot water to warm. $M(t)$ is measured in degrees Celsius, and t is the number of minutes after the bottle is removed from the refrigerator and placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. Students were also given that $M(t) < 40$ for all values of t.

Part a:

A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ (shown below) was given. The task was to sketch the solution curve through the point (0, 5).



Part b:

Students were asked to approximate $M(2)$, the temperature of the milk at time $t = 2$, using the line tangent to the graph of M at $t = 0$.

Part c:

Students had to write an expression for $\frac{d^2M}{dt^2}$ in terms of M and use this expression to determine whether the approximation found in part b is an overestimate or an underestimate for the actual value of $M(2)$. A reason for this answer had to be provided.

Part d:

Students had to use separation of variables to find an expression for $M(t)$, the particular solution for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Comments on student responses and scoring guidelines:

Part a worth 1 point

For 1 point, the curve had to contain the point (0, 5), extend reasonably close to the right side of the slope field, have no obvious conflicts with the given slope lines, and stay below the line $M = 40$. Most students presented an acceptable sketch. A slight "wobble" was accepted as was a sketch that stopped a couple slope lines from the right. Most unacceptable sketches were so because of touching or resting on the line $M = 40$. Some sketches that tried to show an asymptote, but failed to rise above $M = 35$ did not earn this point.

Part b: worth 2 points

The first point was earned for the slope of the line tangent to the graph of M at the point where $t = 0$. Use of $\frac{1}{4}(40 - 5)$ computed this slope, although no work needed to be shown. In fact, $\frac{35}{4}(2) + 5$ by itself earned both points. The second point was for the approximation. Some students computed slope by incorrectly using $\frac{1}{4}(40 - 0) = 10$ or $\frac{1}{4}(40 - 2) = \frac{19}{2}$. This did not earn the first point, but was eligible for the second point with a consistent answer for the approximation. Some students launched straight into the tangent line by showing $y - 5 = \frac{35}{4}(2 - 0)$ which was acceptable for both points provided this was solved for y. However, students who showed only $y - 5 = 10(2 - 0) \longrightarrow y = 25$ were not eligible for the second point because 10 had not been declared as the slope from $\frac{1}{4}(40 - M)$.

Part c: worth 2 points

The first point was earned for a correct expression for $\frac{d^2M}{dt^2}$ which was sometimes seen as $-\frac{1}{4}\frac{1}{4}(40 - M)$. The expression $-\frac{1}{4}\frac{dM}{dt}$ was not acceptable as this is not an expression in terms of M. In order to be eligible to earn the second (reason) point for stating "overestimate" any errors in computation of $\frac{d^2M}{dt^2}$ had to result in a linear function of M. This allowed students who left off the minus sign and presented

something like $\frac{1}{16}(40-M)$ to be eligible for reasoning that the approximation in part b is an underestimate. Correct reasoning from a correct second derivative could state that M is concave down, that $\frac{d^2M}{dt^2}<0$ or that $\frac{dM}{dt}$ is decreasing. Reasoning that appealed to the sign of $\frac{d^2M}{dt^2}$ at a specific point did not earn the second point for this local argument.

<u>Part d:</u> worth 4 points

The first point was earned for a separation of variables, and many students correctly presented one of the forms $\frac{dM}{40-M}=\frac{1}{4}dt$, $4\frac{dM}{40-M}=dt$, $\frac{dM}{10-.25M}=dt$ or an integral of one of these. An incorrect separation such as $(40-M)dM=\frac{1}{4}dt$ took the student out of earning any of the 4 points in part d. The second point was for the antiderivatives. An incorrect form of $\pm k\ln(40-M)$ for non-zero k kept the student eligible for the third point. This third point was for substituting 0 for t and 5 for M in an equation with a correct $+C$. Some students included the $+C$ with the antiderivatives, but went on to exponentiate and/or do some subsequent algebra before the substitution. Some students mishandled this as in $e^{-\ln(40-M)}=e^{\frac{1}{4}t}+C$ and earned neither the third nor the fourth (answer) point. Some students with a correct $e^{-\ln(40-M)}=e^{\frac{1}{4}t+C}$ could easily get the third point by presenting $e^{-\ln(35)}=e^{0+C}$, but had difficulty handling the minus sign in the exponent on the left side when trying to solve for M, perhaps assuming that $e^{-\ln(40-M)}=-\ln(40-M)$.

Readers had to look carefully at work and final results in solving for M. Using the correct $e^{-\ln(40-M)}=e^{\frac{1}{4}t+C}$ shown above, students should conclude that $C=-\ln(35)$. This results in the somewhat awkward work:

$$\begin{aligned} e^{-\ln(40-M)} &= e^{\frac{1}{4}t-\ln(35)} \\ \frac{1}{40-M} &= \frac{1}{35}e^{\frac{1}{4}t} \\ 40-M &= 35e^{-\frac{1}{4}t} \\ M &= -35e^{-\frac{1}{4}t}+40 \end{aligned}$$

Many students did not complete this work correctly, not dealing well with proper use of the minus sign.

Another awkward solution arose from the antiderivatives of $\frac{dM}{10-.25M}=dt$. The work begins as follows:

$$\begin{aligned} -4\ln\left(10-\frac{M}{4}\right) &= t+C \\ \frac{1}{\left(10-\frac{M}{4}\right)^4} &= e^{t+C} \end{aligned}$$

Most students in this situation did not correctly solve for M. Students whose algebra was fairly good could end up with solutions such as $M = 40 - e^{-\frac{1}{4}t + \ln(35)}$ or $M = 40 - e^{-\frac{1}{4}(t - 4\ln(35))}$ which are both correct. More complicated versions of this type of solution were presented, and readers had to carefully check to see if these forms of an expression for M might have mishandled a minus sign or a 4. (See **Observations and recommendations for teachers** #5)

Observations and recommendations for teachers:

(1) When sketching a curve given its derivative, a slope field, and an initial condition, two things are of utmost importance: include the initial condition point and follow the slope field. The existence of an asymptote in the case of this question was implied strongly in the slope field. It can be difficult for some to sketch this smoothly (as this reader can verify), but unwanted marks should be erased or clarified with a note accompanying the sketch.

(2) A fundamental fact often presented when the idea of a derivative is first introduced is that $\frac{dy}{dx}$ gives the slope of a line tangent to a curve given by $y = f(x)$. Thus, use of the given $\frac{dM}{dt}$ will give the slopes of lines tangent to the graph of M. At any given point on this graph (in the case of part b of this question the point (0, 5)) the slope of a line tangent to the graph of M can be computed. With this slope, the equation of this tangent line can be written and then used to approximate the value of M at any other point on the graph. If concavity of M either up or down can be determined on an interval containing this point, then the approximation from the tangent line is an underestimate or an overestimate, respectively, of the actual value of M. This is an idea that can be easily visualized and practiced in class.

(3) Students should not simplify on the AP Calculus Exam. In part b, $\frac{35}{4}(2) + 5$ was simplified incorrectly a number of times. In part c, $\frac{1}{4} \bullet \frac{1}{4}$ was sometimes changed to $\frac{1}{8}$. Simplification is not required on the FRQs.

(4) Students should read the questions carefully; and before deciding that work was done, read them one more time to make sure that everything asked for was answered. In part c $\frac{d^2M}{dt^2}$ was asked for as a linear function of M. Many students stopped work with the presentation of $-\frac{1}{4}\frac{dM}{dt}$. Students were not eligible for the reason point without a linear function of M from which to argue.

(5) A possible reason for low scores on this question was the inability of many students to earn more than one or two points on part d. This reader would recommend substituting the initial condition as soon as possible after antiderivative work. This quickly results in $\frac{dM}{40-M}=\frac{1}{4}dt \longrightarrow -\ln(40-M)=\frac{1}{4}t+C \longrightarrow -\ln(35)=C$. Before exponentiating, multiplying both sides of $-\ln(40-M)=\frac{1}{4}t-\ln(35)$ by -1 yields $40-M=e^{-\frac{1}{4}t+\ln(35)}$ from which solving for M proceeds easily. Multiplying by -1 makes the log term easier to deal with when exponentiating.

(6) Students should practice with a simple derivative such as $\frac{dy}{dx}=x$ or $\frac{dy}{dx}=\frac{x}{y}$ or $\frac{dy}{dx}=\frac{x^2}{y^3}$ until the steps needed to solve using separation of variables are clear and substitution of the initial condition has been practiced. Then move on to functions involving more complicated antiderivatives such as $\frac{dy}{dx}=\sqrt{1-y^2}$ or $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{x}$ or $\frac{dy}{dx}=\frac{20-y}{x}$, the latter involving use of logs on both sides of the equation with the antiderivatives.

(7) Simplification is *NOT* required on the FRQs. Students should certainly know such things as $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, $6\frac{\sqrt{2}}{\sqrt{3}}=\frac{6\sqrt{6}}{3}=2\sqrt{6}$, $e^{\ln(35)}=35$ and $e^{-\ln(35)}=\frac{1}{35}$, especially since these might be presented as final answer choices on the multiple choice section of the exam. But subsequent simplification efforts on a high stakes exam can often result in minor errors, costing students valuable points on the FRQs.