Problem Overview:

Students were asked to consider the curve defined by the equation $6xy = 2 + y^3$.

Part a:

Students had to show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.

Part b:

This part of the question required finding the coordinates of a point on the curve for which a line tangent to the curve is horizontal, or explain why no such point exists.

Part c:

This part of the question required finding the coordinates of a point on the curve for which a line tangent to the curve is vertical, or explain why no such point exists.

Part d:

Students were given that a particle is moving along the curve as well as the additional information that at the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. Students were asked to find the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant.

Comments on student responses and scoring guidelines:

Part a worth 2 points

The first point was earned for the implicit differentiation of the equation $6xy = 2 + y^3$ with respect to x. The equation $6x\dfrac{dy}{dx} + 6y = 3y^2\dfrac{dy}{dx}$ had to be correct in order to earn this first point. The second point was for using the result of this implicit differentiation to arrive at either $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$ or, with no subsequent errors, $2y = \dfrac{dy}{dx}\left(y^2 - 2x\right)$. Many students earned the first point, but had difficulties solving the equation for $\dfrac{dy}{dx}$. There were also difficulties communicating with proper notation in order to earn the second point, as in presenting something like $\dfrac{6y}{3y^2 - 6x} \div 3 = \dfrac{2y}{y^2 - 2x}$.

Part b: worth 2 points

The first point was for evidence of setting $\dfrac{dy}{dx} = 0$. All of the following were acceptable if set equal to 0: $2y,\ y,\ y',\ \dfrac{dy}{dx}$ or $\dfrac{2y}{y^2 - 2x}$. For the second point, an answer with reason had to be provided and involved more work. A simple way to show that there is no point of horizontal tangency is to substitute $y = 0$ into the original equation, resulting in $6x \bullet 0 = 2 + 0$, for which there is no solution. Many students did not earn this second point because of failure to appeal to the original equation defining the curve.

Part c: worth 3 points

The first point was earned for connecting a vertical tangent to the fact that the denominator of $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$ could not equal 0. The second point was for a substitution that would lead to the answer. This could be substituting $y = \sqrt{2x}$ into the original equation, but few students chose this method and usually did not explain why they were using $y = \sqrt{2x}$ rather than $y = -\sqrt{2x}$. Perhaps a bit more difficult to resolve is substituting $\dfrac{2 + y^3}{6y}$ for x into $y^2 - 2x = 0$. However, from $\dfrac{2 + y^3}{6y}$ one could compute $\dfrac{dx}{dy} = \dfrac{y^3 - 1}{3y^2}$ and see from this that we have $\dfrac{dx}{dy} = 0$ where $y = 1$. The best approach is to substitute $x = \dfrac{y^2}{2}$, determined from $y^2 - 2x = 0$, into the original equation. (Note that in all these approaches, the original equation defining the curve must to be used.)

The third point was for $\left(\dfrac{1}{2}, 1\right)$. These coordinates could be presented separately if labeled.

Part d: worth 2 points

The first point was earned for evidence of correct implicit differentiation of $6xy = 2 + y^3$ with respect to t. Presenting any of the terms $6y\frac{dx}{dt}$, $6x\frac{dy}{dt}$ or $3y^2\frac{dy}{dt}$ earned this first point. The second point was earned for the answer $-\frac{8}{9}$. (References to units were ignored.) The most common solution used substitution of $\frac{dx}{dt} = \frac{2}{3}$ and $(x,\ y) = \left(\frac{1}{2},\ -2\right)$ into $6x\frac{dy}{dt} + 6y\frac{dx}{dt} = 3y^2\frac{dy}{dt}$.

An alternate solution was sometimes seen, beginning with $\frac{dy}{dt} = \frac{dy}{dx}\bullet\frac{dx}{dt}$. The value of $\frac{dy}{dx}$ can be computed by substituting $(x,\ y) = \left(\frac{1}{2},\ -2\right)$ into $\frac{2y}{y^2 - 2x}$. The result is $-\frac{4}{3}$, leading to $\frac{dy}{dt} = -\frac{4}{3}\bullet\frac{2}{3}$.

Observations and recommendations for teachers:

(1) When a curve is defined implicitly in terms of x and $y(x)$, computing $\frac{dy}{dx}$ involves two skills. First, the derivative with respect to x must be calculated, term by term. The second skill is “algebraic,” requiring solving for $\frac{dy}{dx}$. If we substitute A for “answer” wherever $\frac{dy}{dx}$ appears, then we have:

$$6x\frac{dy}{dx} + 6y = 3y^2\frac{dy}{dx} \longrightarrow 6xA + 6y = 3y^2A$$

If solving for A is a difficulty (and it is for many students), this points to a not uncommon deficit in algebraic facility, namely solving literal equations. There is nothing wrong with spending some class time reinforcing and practicing this skill.

(2) A horizontal tangent to a curve exists where $\frac{dy}{dx} = 0$. In this question, that means that the numerator of the expression for $\frac{dy}{dx}$ must be 0. Here we get $y = 0$ which should be immediately substituted into the equation defining the curve in order to any find value(s) of x for points which lie on the curve. If this results in a falsehood such as $0 = 2$, then there is no point of horizontal tangency.

(3) The search for a vertical tangent line begins by looking for points where $\frac{dy}{dx}$ does not exist. This can occur when the denominator of the expression for $\frac{dy}{dx}$ is 0. In this question, we do not get a specific value for either x or y. Rather, we get a relationship between x and y. Again, information from this relationship must be substituted into the original equation defining the curve. A number of students did not refer to the original equation successfully, if at all.

Time permitting, further exploration of situations in which both numerator and denominator are equal to 0 can be insightful. One source of information and examples with which to begin this study can be found in *Calculus: Dynamic Mathematics, Volume One* by Chuck Garner.

(4) Recognizing in part d that the original equation defining the curve did not have to be used saved time (unless a diligent student just wanted to verify that the point $\left(\frac{1}{2}, -2\right)$ is indeed on the curve). Recognizing that $\frac{dy}{dt} = \frac{dy}{dx} \bullet \frac{dx}{dt}$ also saved some work. This worked well in this part of the question because an expression for $\frac{dy}{dx}$ was available. Computing the derivative term by term of $6xy = 2 + y^3$ with respect to t is more involved and also requires use of the product rule.

(5) Note that in two parts of this question, the original equation defining the curve had to be used. From the number of student responses not recognizing this, it is clear that practice with this type of question is needed.