2023 AB5 Marshall Ransom, Georgia Southern University

Problem Overview:

The functions f and g are twice differentiable. Some values of f and g and their derivatives were given in the table shown below.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	–8	3	6
$g(x)$	1	2	–3	0
$g'(x)$	5	4	2	8

Part a:

The function h was defined by $h(x) = f\left(g(x)\right)$. Students were asked to find $h'(7)$ and show the work leading to the answer.

Part b:

k is a differentiable function such that $k'(x) = \left(f(x)\right)^2 \bullet g(x)$. This part of the question asked whether the graph of k is concave up or down at the point where $x = 4$ and for students to give a reason for the answer.

Part c:

The function m is given as $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Students were asked to find $m(2)$ and show the work that leads to this answer.

Part d:

Students had to determine whether the function m defined in part c is increasing, decreasing or neither at the point where $x = 2$ and justify the answer.

Comments on student responses and scoring guidelines:

Part a worth 2 points

The first point was earned for using the chain rule, and the second point was earned for the answer with supporting work. The first point was earned for either $f'(g(x))\bullet g'(x)$ or $f'(g(7))\bullet g'(7)$. A not uncommon error seen was what is referred to at the reading as a "linkage" error as in $h'(x) = f'(g(7))\bullet g'(7)$. If the first point was not earned, the second point could be earned only by presentation of either $f'(0)\bullet 8 = 12$ or $\frac{3}{2}\bullet 8$. A response of 12 with no supporting work earned neither point.

Part b: worth 3 points

The first point was for evidence of either the chain rule or the product rule in computing k''. Examples of acceptable, although erroneous, expressions were detailed in the scoring guidelines, each showing just one error such as:

$2f(x)g(x) + (f(x))^2 g'(x)$ – product rule but chain error

$2f'(x)g(x) + (f(x))^2 g'(x)$ – product rule but chain error

$f'(x)g(x) + (f(x))^2 g'(x)$ – product rule but chain error

$2f(x)f'(x)g'(x)$ – chain rule but product error

These types of expressions earned the first point but were not eligible to earn the second point for the value $k''(4) = -40$. Another way to not earn the second point was a linkage error such as $k''(x) = -40$.

Students seemed to have more difficulty using the chain rule to find the derivative of $(f(x))^2$ in part b than in finding the derivative of $f(g(x))$ in part a. The third point was earned for an answer and reason consistent with any non-zero declaration of $k''(4)$. A clear and sufficient answer for the third point is k is "concave down at $x = 4$ because $k''(x)$ is negative." Ambiguous language such as "it is negative" or "the acceleration is negative" were not acceptable reasons.

Part c: worth 1 point

This one point was earned for work that arrived at the answer 37. There were errors in using the fundamental theorem such as $\int_0^x f'(t)\,dt = f'(x) - f'(2)$ or $\int_0^x f'(t)\,dt = f(x)$. Again, some students exhibited linkage errors such as $m(2) = 5(2)^3 + f(x) - f(2)$ or $m(x) = 5x^3 + \int_0^2 f'(t)\,dt$.

Part d: worth 3 points

In order to determine increasing or decreasing for function m, students were expected to show evidence of considering m'. This earned the first of 3 points in this part of the question. Acceptable work for the second point had to accompany the answer of 52 as in $m'(2) = 15(2)^3 + f'(2)$ or $m'(2) = 60 + (-8)$. An unsupported answer of 52 did not earn the second point. The third point was for an answer with correct reasoning supporting any declared value of $m'(2)$. The statement "$m'(2)$ =52, therefore m is increasing at $x = 2$" was not acceptable for the third point. The reasoning had to appeal to the sign of $m'(2)$. Linkage errors such as $m'(x) = 15x^2 + f'(x) = 15(2)^2 + f'(2)$ and $m'(2) = 15x^2 + f'(2)$ occurred.

Observations and recommendations for teachers:

(1) Computing the derivative of $f\left(g(x)\right)$ involves a basic use of the chain rule, sometimes taught using language such as "get the derivative of the first (or "outside") function, then multiply by the derivative of the inside function." Finding the derivative of something like $\left(f(x)\right)^2$ means that the "outside" function, the squaring function, must be recognized first. The first step here before multiplying by the derivative of the inside function is to apply the power rule. When dealing with trig functions, this power rule necessity can be obscured by the common notation of something like $\sin^3\left(g(x)\right)$ rather than $\left(\sin\left(g(x)\right)\right)^3$.

(2) Students should be given feedback on and warned about "linkage" errors, perhaps defining a function by a single value as in $h'(x) = f'\left(g(7)\right) \bullet g'(7)$. If the variable is replaced by a specific value, it should be replaced everywhere by that value. The value of the variable cannot appear "gradually" during student work on the AP Exam.

(3) The evaluation part of the Fundamental Theorem states that $\int_a^b f(t)\,dt = A(b) - A(a)$ where $A'(x) = f(x)$. In the case of $\int_0^2 f'(t)\,dt$, the antiderivative, $A(x)$, needed is $f(x)$ itself. Many students did not seem prepared to see this.

(4) Linkage errors were all too common in student work on this question. This could cost a student one or more valuable points.