2023 AB/BC1 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given that the differentiable function f models the rate at which gasoline is pumped into a tank. $f(t)$ is measured in gallons per second and t is time in seconds since the pumping began. Some values of $f(t)$ are given in the table below.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gals. per sec.)	0	0.1	0.15	0.1	0.05	0

Part a:

Students had to interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem, using correct units.

Students also had to approximate the value of this integral using a right Riemann sum and the intervals [60, 90], [90, 120] and [120, 135].

Part b:

Students were asked if there must be a value c with $60 < c < 120$ for which $f'(c) = 0$ and to justify the answer.

Part c:

The rate of flow in gallons per second could also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$.

Students had to use this model to find the average rate of flow over the interval $0 \le t \le 150$ and to show the setup for any calculations.

Part d:

Students had to use the model $g(t)$ given in part c to find the value of $g'(140)$ and interpret the meaning of this value in the context of the problem.

<u>Comments on student responses and scoring guidelines:</u>

<u>Part a</u> worth 3 points

The first point was earned for the interpretation of the definite integral. A response needed to include that this was the total number of gallons (units) pumped into the tank and that this took place between times $t = 60$ seconds and $t = 135$ seconds. Responses such as "this is the total number of gallons in the tank between $t = 60$ seconds and $t = 135$" seconds did not earn this point.

The second point was earned for work showing a form of a correct Riemann sum (the sum of products of function values and differences). Responses with at least five of the six needed numbers correct earned this point. In order to earn the third point, all had to be correct. Minimal responses such as $0.15(30) + 0.1(30) + 0.05(15)$ or $f(90)(90-60) + f(120)(120-90) + f(135)(135-120) = 8.25$ earned both of these points. Responses not showing enough work to indicate the computation of a Riemann sum such as $4.5 + 3.0 + 0.75$ earned the third point but not the second. A completely correct left Riemann sum earned one of the last two points.

<u>Part b:</u> worth 2 points

The first point was for showing the difference $f(120) - f(60) = 0$. This could be shown as $0.1 - 0.1 = 0$, $f(120) = f(60)$ or as a difference in the numerator of a quotient. The second point was earned for establishing the hypotheses of the Mean Value Theorem and for the answer. Since f was given as differentiable, this required stating somehow that this differentiability implied continuity. A named theorem could be MVT or Rolle's Theorem, but Intermediate Value Theorem could not be named.

<u>Part c:</u> worth 2 points

The first point was earned for showing the average value formula correctly, most simply by $\dfrac{1}{150}\displaystyle\int_0^{150} g(t)\,dt$.

Writing the entire expression for $g(t) = \left(\dfrac{t}{500}\right)\cos\left(\left(\dfrac{t}{120}\right)^2\right)$ risked a copy error. The second point was earned for the correct answer, 0.0959967 or this value rounded or truncated to three decimal places. Incorrect communication as in $\displaystyle\int_0^{150} g(t)\,dt = \dfrac{14.399504}{150} = 0.0959967$ earned one of these two points while $\displaystyle\int_0^{150} g(t)\,dt = 0.0959967$ earned neither point.

<u>Part d:</u> worth 2 points

The first point was earned for the value $g'(140) = \dfrac{1}{500}\cos\left(\dfrac{49}{36}\right) - \dfrac{49}{9000}\sin\left(\dfrac{49}{36}\right) \approx -0.004908$ or this decimal value correctly rounded or truncated to three places after the point. The second point was for the interpretation of this value. This had to include that this was the rate that the declared value of $g'(140)$ at $t = 140$ is changing. Since this value is negative, language such as "the rate is decreasing at a rate of -0.005" was not acceptable because the double negative in that phrasing implies $+0.005$.

Observations and recommendations for teachers:

(1) Computation of a Riemann sum in order to approximate the value of a definite integral requires some practice, especially using intervals of different widths. The AP Calculus Exam may ask for a left, right or midpoint Riemann sum. Students should be reminded that simplification of an arithmetic setup is not required on the FRQ portion of the exam. Interpretation (verbally describing the meaning of the value in the context of the problem) needs to be practiced as well. Good examples of proper wording can be found in past AP exams and other AP preparation materials.

(2) The wording of the problem posed in part b of this question should signal that either the MVT or IVT is needed since the value of interest is between two endpoints of an interval of a differentiable function, $f(t)$. Since continuity is required in order to apply either of these theorems, it must be stated that the given differentiability implies that continuity. No values of $f'(t)$ are given, so it should be obvious that IVT cannot be applied to $f'(t)$, which leaves the MVT. The MVT asserts that the average rate of change on an interval must be equivalent to the derivative somewhere on that interval. Showing some work first requires computing the change in the values of $f(t)$. Since this change is 0, the predicted $f'(t) = 0$ for some value of t with $60 < t < 120$. Students realizing this and providing evidence of correct work should be careful to read the question in part b to realize that an answer is needed.

(3) The average value of a function on an interval [a, b] is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$, expected to be known by AP students. The first work needed to be shown is this setup. If a function is given by a named explicit expression, it is best to use the function name as the integrand rather than risking a copy error in writing the entire expression. Students on a calculator active question sometimes have access to a CAS, and this exact answer is acceptable. But once again, this risks the possibility of a copy error.

(4) In part d of this question, the value of $g'(140)$ had to be presented. This is a reminder that explicit expressions for functions or complete exact values do not have to be presented if the function has a name or if the value can be rounded or truncated to three accurate digits after the point. Showing additional digits after the first three is fine because the only thing that matters is that the first three are correct.

(5) What is requested in this question is quite typical on the AP Calculus Exam. Past exam questions can provide guidance: See 2013 AB/BC1, 2013AB/BC3, 2014AB/BC1, 2015AB/BC3, 2016AB/BC1, 2017AB/BC1, 2018AB2.