2022 BC6 Storie Atkins, Columbus High School and Marshall Ransom, Georgia Southern University

Problem Overview:

The function, f, was given, defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

Part a:

Students were asked to use the ratio test to find the interval of convergence of the power series for f.

Part b:

Students were asked to show and justify that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$.

Part c:

Students needed to write the first four nonzero terms and general term for an infinite series representing $f'(x)$.

Part d:

Using their answer from part c, students needed to evaluate $f'(x)$ at $x = \frac{1}{6}$.

Comments on student responses and scoring guidelines:

Part a worth 4 points

To earn the first point, students had to present a correct ratio, with or without absolute values. We note that:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}}\right| = \lim_{n\to\infty}\left|x^2\left(\frac{2n+1}{2n+3}\right)\right| = |x^2|$$

Students who wrote $\frac{2n+1}{2n+2}$ were eligible for the second and third points, but not the fourth. Subsequent good calculus could earn points despite an arithmetic error appearing in the ratio setup.

For the second point, students needed to include limit notation and identify the interior of the interval of convergence, writing it in either interval or inequality notation. Somewhere, there needed to be a limit with the ratio to earn the second point. Even poor notation, such as $\underset{n\to\infty}{\mathcal{L}}$, was accepted. For the interval, it was not sufficient to write $|x| < 1$. The correct interior interval could be found anywhere in part a, even as a circled or boxed final answer. Students needed the correct interval to be eligible for the third point of part a, so readers were instructed to look for the interval anywhere in student work.

Students needed to consider both endpoints to be eligible for the third point of part a. Students had to use ±1 in evaluating the general term or by crunching out a few terms. In evaluating the general term at $x = 1$, students had to present $\frac{(-1)^n}{2n+1}$, not $\frac{1}{2n+1}$. If students had an x or x^3 along with a correct ratio, the response was eligible for this point, but not the second or fourth points. Students who presented the incorrect interval $0 < x < 1$ or $x < 1$ with final interval [0, 1] could earn the third point if the general term was evaluated at $x = 1$.

For the final point, students had to present the correct interval, [-1, 1], with justification. The only accepted justification had to refer to the alternating series test. Students could show the two conditions of the test, decreasing and limit of 0 for the absolute value of the terms, or refer to the alternating series test, AST. To earn the fourth point, the student needed to use the correct endpoints in the correct general term, and the only correct justification was AST, referenced as the abbreviation, the words, or the conditions of the test. Students did not need to earn the third point to be eligible for the fourth point.

<u>Part b</u> worth 2 points

In part b, students needed all three, bulleted statements in the scoring notes to earn the second point. This was a high bar for justification. The response had to (1) have earned the first point by correctly using $x = \frac{1}{2}$ in the second term, and only the second term, (2) state the series is alternating with terms decreasing to zero, and (3) state the error inequality, $\frac{1}{24} < \frac{1}{10}$. This point was awarded even if the inequality presented was $-\frac{1}{24} < \frac{1}{10}$; however, the second point was not awarded if the response stated that the error = $\frac{1}{24}$.

<u>Part c</u> worth 2 points

For part c, students earned the first point by writing the first four nonzero terms for $f'(x)$, and the second point was earned by presenting the general term. To earn the general term point, parentheses were required for $(-1)^n$. Alternate versions of the general term were accepted, and readers were expected to use values of *n* to see if the reported general term generated the first four correct terms. The general term could be presented using summation notation as follows: $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ or $\sum_{n=1}^{\infty}(-1)^{n+1} x^{2n-2}$.

<u>Part d</u> worth 1 point

For part d, the only accepted answer was 36/37 or an unsimplified version it. In order to be eligible for this point, students had to present an answer in part c that was geometric. If students provided an incorrect, geometric response in part c, readers were allowed to read with them for earning this point. This point was not earned if the response reported an approximation, the sum of the first four terms, in place of the exact sum. Many students did a lot of work to add those four terms together, which only led to an unacceptable approximation.

<u>Observations and recommendations for teachers:</u>

(1) The alternating series test and the ratio tests are in the BC curriculum. Frequently, these are needed on the BC Exam FRQs in order to answer and justify answers.

(2) Using the ratio test requires the setup of a ratio, which usually in itself earns a point on the exam.

(3) Up through the 2022 AP Calculus Exam, using the alternating series test required either stating the conditions of decreasing and limit of the absolute values of the terms being zero or just reporting "AST." As expectations increase for the use of hypotheses in the application of theorems, it might be best to emphasize when teaching that merely writing "AST" is not sufficient. This also provides a learning opportunity to examine a few series for which the limit of the absolute values of the terms is 0, but the series does not converge because of a lack of decreasing term-by-term. An example is given by $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{16}+\frac{1}{5}-\frac{1}{32}+\cdots$ in which (a) the limit of the absolute values of the terms is 0, (b) but term-by-term decreasing is not satisfied. The positive terms form the divergent harmonic series, but the negative terms have a sum of -1.

(4) When analyzing convergence at the endpoints of an interval of convergence, it is extremely important with an alternating series to use $(-1)^p$ with the -1 in parentheses (along with the relevant exponent).

(5) When using *m* terms of an alternating series to approximate the value of a function, the error is less than the absolute value of term $m+1$. In this question, the first such term with an absolute value less than $\frac{1}{10}$ is the second term. Since the value of the first term of the series at $x=\frac{1}{2}$ is $\frac{1}{2}$, we have the following: $\left|f\left(\frac{1}{2}\right)-\frac{1}{2}\right|<\frac{1}{24}<\frac{1}{10}$. Students should be familiar with the alternating series error bound and experienced in using it.

(6) Term-by-term differentiation of the series for $f(x)$ produces a series for f'. In particular, the given general term can be differentiated to yield a general term for the series for f'.

(7) In this question, the series for f' is geometric, and students should recognize this as such. Substituting $\frac{1}{6}$ for *x* into this series yields an infinite geometric series with ratio $r=-\frac{1}{36}$, and we have $|r|<1$. Therefore, the value $f\left(\frac{1}{6}\right)=\dfrac{\textit{first term}}{1-r}=\dfrac{1}{1-\left(-\frac{1}{36}\right)}$. (Remind students: DO NOT BOTHER TO SIMPLIFY!!)

(8) When first using geometric series, a variable can be introduced, and an interior interval of convergence can be determined using $|r|<1$. This can easily lead to calculus and substitution methods before getting into Taylor series. For example, $\dfrac{1}{1+x^2}=\dfrac{\textit{first term}}{1-r}=1-x^2+x^4-x^6+\cdots$ Term-by-term integration yields $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$ where $f(x)$ is $\arctan(x)$, the series used in this question.