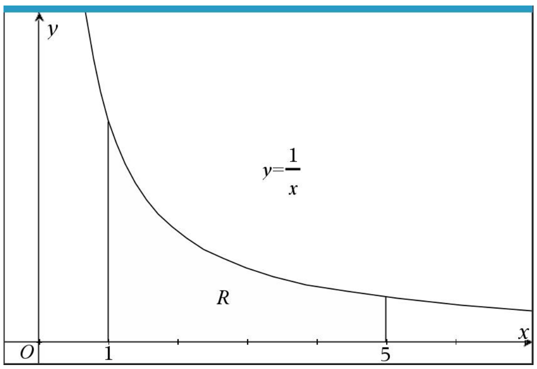
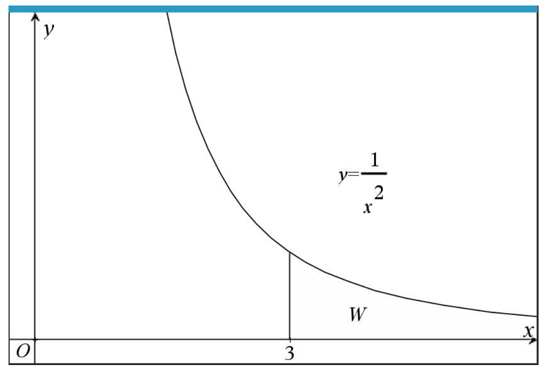
2022 BC-5 Dennis Wilson, Landmark Christian School

Problem Overview





The students were given the graphs of the functions $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ in two separate figures as shown above. The first figure contained the region R bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. The second figure contained the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis lying to the right of the vertical line $x = 3$.

Part a

Students were asked to find the area of the region R shown in the first figure.

Part b

Students were told that the region R was the base of a solid. The cross sections of this solid perpendicular to the x-axis, at each x, had an area given by $xe^{x/5}$. Student were asked to find the volume of this solid.

Part c

Students were asked to find the volume generated when the unbounded region W was rotated around the x-axis.

Comments on Student Responses and Scoring Guidelines

Part a: worth 2 points

The first point was awarded for an integral expression equivalent to the area of region R. This was most often expressed as $\int_1^5 \frac{1}{x}\, dx$.
The second point was awarded for the answer of $\ln 5$. Bald answers of $\ln 5$ received no points at all. The expression $\ln 5 - \ln 1$ also received the second point. Student responses which did not include

any integral could only earn the second point with this expression of the difference.

Part b: worth 4 points

The first point for part b was earned for any nonzero constant multiple of a definite integral equivalent to the volume of the solid. These answers would be of the form $c\int_1^5 xe^{x/5}\,dx$ where $c \neq 0$. If students used a value of c other than 1, then they were not able to earn the fourth point which is for the answer.

The second and third points were for the evaluation of the definite integral using integration by parts. Students earned the second point by applying integration by parts. The application of integration by parts could be demonstrated by students in multiple ways. One way was if students labeled factors of the integrand as u and dv. The students would earn this point even if they chose their factors incorrectly or if their computations of du and v were incorrect. A second way students could earn this point was using a table. The columns of the table did not need to be labeled but the columns needed to begin with x and $e^{x/5}$. Students could earn the second point implicitly simply by writing the expression $5xe^{x/5} - \int 5e^{x/5}\,dx$. The third point was explicitly awarded for this expression. Students who did not show work leading to an expression that contained an error could not earn either point. The limits of integration for the definite integral did not need to be present anywhere to earn the first and second point.

The fourth point was for the correct answer. Students could only earn this point if they had earned the previous three points for part b.

Part c: worth 3 points

The first point of part c was earned by presenting an improper integral equal to the volume of the solid. The definite integral could presented with infinity as a limit of integration,

$$\pi \int_3^\infty \left(\frac{1}{x^2}\right)^2 dx,$$

or it could be presented with a limit,

$$\lim_{b\to\infty} \pi \int_3^b \left(\frac{1}{x^2}\right)^2 dx.$$

If students forgot to include the constant of π or used any other nonzero constant in its place, they would still earn this point. The error would forfeit the third point.

The second point was earned for a correct antiderivative of any integrand of the form $\frac{1}{x^p}$ where p is an integer greater than or equal to two. Many students earned only this second point for part c. They had correctly set up an improper integral for the area of region W instead of the volume of the solid generated by revolving the region.

The third point, which was for the answer, proved the most difficult to earn. In order to earn this point, the correct use of limit notation needed to be present in the problem. The use of the limit could appear late but once it appeared, it could not disappear until the limit was evaluated. Students also could not earn this point if their response included infinity substituted in as a numerical value.

Observations and Recommendations for Teachers

(1) There were many bald answers of $\ln 5$ for part a. Students who learn transcendental functions in the later part of the year are frequently taught(and correctly so) that the natural logarithm of some value is the area bound by the graph of $y = \frac{1}{x}$ between 1 and that value. I wondered if it were these students who answered the question with bald answers. If this is the case or not, students should be told to provide supporting work for their answers to free-response questions. It is only in the rarest of circumstance that bald answer receive any points. In most of these cases, the question informs the student that no computations are required.

(2) Many students were unable to earn any points for part b. This arose because students tried to incorporate the equation for the function which bound region R, $y = \frac{1}{x}$, into their integral. The majority of incorrect responses were of the form

$$\int \left(\frac{1}{x}\right)(xe^{x/5})\,dx = \int e^{x/5}\,dx$$

which does not require integration by parts. All volume questions on the AP Calculus Exam involve an integral of the form $\int A(x)\,dx$ where $A(x)$ is the area of the perpendicular cross section. This is even true for volumes of revolution. Students should be taught that volume is just the accumulation of area, and for the AP Calculus Exam, the accumulation of the area of the cross section. As the problem stated that the rectangle has an area of $xe^{x/5}$, students should have known to just use the given expression for their integrand. A useful way to emphasize this concept to students is by having them compute volumes when the function for the area is given and does not need to be found. This also provides an important opportunity to provide students with much needed practice in computing the values of definite integrals.

(3) The method of tabular integration by parts is an important technique which students who study mathematics beyond the AP Calculus classroom should know. This method does not save much time in the context of an AP exam and some AP edition textbooks do not even cover the method. This is unfortunate. The instruction of this tabular method provides opportunities for students to deepen their understanding of calculus and to sharpen their skills. This method applies easily to integrals of the form $\int f(x)g(x)\,dx$ where $f(x)$ is a polynomial and where the antiderivatives of $g(x)$ can be easily computed. Students can recognize that repeated differentiation of a polynomial will eventually lead to 0. The examples of $g(x)$ that arise most frequently are compositions of exponential or sinusoidal functions with linear functions, such as e^{2x}, $\sin(3x-\pi)$, or $\cos(x/2)$. Here students can be instructed on avoiding substitution for a linear function by simply dividing the antiderivative by the slope of the linear function. That is $\int f'(mx+b)\,dx = \frac{1}{m}f(mx+b)+C$. And so $\int e^{2x}\,dx = \frac{1}{2}e^{2x}+C$ and $\int \cos(x/2)\,dx = 2\cos(x/2)+C$.

(4) The fifth question on the AP Calculus BC exam has generally become a veritable potpourri of topics specific to the BC curriculum. Students should be aware of which topics are purely BC, such as improper integrals and integration by parts which this problem included. Other BC only

topics that have appeared in the fifth question involve but are not limited to Euler's Method, Partial Fractions, and series. In preparation for the exam, the typical content of questions should be discussed with students so they may be successful. While the correct application of calculus in part b would have prevented students bypassing integration by parts, the knowledge that the question was BC specific could have served as a reminder when considering the scope of the entire question.

(5) Students need to exercise caution when writing expressions which involve infinity. Many students responded to part c with equations such as

$$\lim_{t\to\infty}\left(\frac{1}{t^3}-\frac{1}{3^3}\right)=\left(\frac{1}{\infty^3}-\frac{1}{27}\right)=\left(0-\frac{1}{27}\right).$$

While substituting infinity as a value in an algebraic expression is helpful for student reasoning, it is mathematically incorrect. Students who need to do this should be encouraged to perform their written reasoning off to the side, outside the body of the mathematical work. In general, a student will lose the answer point if he or she sets an expression involving infinity equal to something else.