2022 BC2 Marshall Ransom, Georgia Southern University

Problem Overview:

Students are given that a particle is moving along a curve in the *xy*-plane with position given by $(x(t),\ y(t))$ for $t>0$. It is also given that $\dfrac{dx}{dt}=\sqrt{1+t^2}$, $\dfrac{dy}{dt}=\ln\left(2+t^2\right)$, and at time $t=4$ the position of the particle is the point (1, 5).

Part a:

Students were asked to find the slope of the line tangent to the path of the particle at time $t=4$.

Part b:

Students were asked to find the speed as well as the acceleration vector of the particle at time $t=4$.

Part c:

Students were asked to find the *y*-coordinate of the particle's position at time $t=6$.

Part d:

Students were asked to find the total distance the particle travels along the curve from time $t=4$ to time $t=6$.

Comments on student responses and scoring guidelines:

Part a worth 1 point

The slope to this curve at $t=4$ is given by $\left.\dfrac{dy/dt}{dx/dt}\right|_{t=4}=\dfrac{\ln(18)}{\sqrt{17}}$. A decimal approximation is 0.701018. Most students recognized this approach to the problem. A common error was the linkage of $\dfrac{dy/dt}{dx/dt}=\dfrac{\ln(18)}{\sqrt{17}}$, showing a ratio of functions equivalent to a constant. (Note: if this type of linkage error appeared in subsequent parts of this question, students were not penalized again). The setup had to be shown, meaning that an unsupported correct decimal approximation did not earn this point. But $\dfrac{\ln(18)}{\sqrt{17}}$ or $\dfrac{\ln(2+4^2)}{\sqrt{1+4^2}}$ with no other work did earn this point, showing evidence that the value of 4 was used for *t* in the given functions.

Part b: worth 3 points

One point was earned for the speed and one for each component of the acceleration vector. The setup for these computations had to be shown. For example, the setup for computing the speed is to calculate the magnitude of the velocity vector which is $\sqrt{(x'(4))^2+(y'(4))^2}$. The speed is $\sqrt{\sqrt{17}^2+\left(\ln(18)\right)^2}$ or 5.035. In alignment with the philosophy applied in part a, the unsupported $\sqrt{\sqrt{17}^2+\left(\ln(18)\right)^2}$ did earn the point for the speed, showing evidence that the value of 4 was used for t in the given functions. However, the second and third points could only be earned if evidence was shown that the value 4 was used for t in the derivative of the given functions (x'' and y'').

Students using explicit expressions for the functions sometimes presented copy or parentheses errors, especially when computing the acceleration vector.

If the components of the acceleration vector were reversed, the response to the problem earned neither of the second or third points. If these components were not presented as an ordered pair, they had to be labeled in order to earn these last two points.

Part c: worth 3 points

The first point was earned for an integrand of $\frac{dy}{dt}$ in either an indefinite or definite integral. Limits on the integral could be incorrect; but if never resolved in subsequent work, the response was ineligible for the third (answer) point. The second point was earned for using the value $y(4)=5$ by adding it to a definite integral. It was not uncommon for students to never arrive at limits on their integrals and/or to fail to properly use the initial condition $y(4)=5$.

Rather than write $\int_4^6\left(\frac{dy}{dt}\right)dt$, some students chose to write $\int\ln(2+t^2)\,dt$, which is perfectly fine. However, some students tried to compute $\int\ln(2+t^2)\,dt$ using integration by parts, which was only rarely successful. Students with CAS-equipped calculators could get this antiderivative symbolically, but sometimes had difficulty copying it correctly.

Part d: worth 2 points

The first point was earned for using the correct integrand $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ in a definite integral. The total distance traveled is the definite integral of the speed. As with part b there were common errors with parentheses and presentation of this integrand. The second point, if the first point had been earned, was for the answer of 12.136. Some students chose other methods, discussed in **Observations and recommendations for teachers** (5) below.

Observations and recommendations for teachers:

(1) Although this is a calculator active problem, "work," meaning setups, had to be shown. Unsupported answers, no matter how easily obtained, often earn no points on the AP Calculus Exam. This applied particularly to parts a and b of this question.

(2) **Use the function names**: The functions $\frac{dx}{dt} = \sqrt{1+t^2}$ and $\frac{dy}{dt} = \ln\left(2+t^2\right)$ were "named" meaning that there is no reason to copy the explicit expressions in the work. Copying and using explicit expressions is not necessary for named functions and often leads to copy, parentheses and other presentation errors which can cost students points.

(3) This is a calculator active question. There should be no need to calculate derivatives or antiderivatives by hand. However, students correctly presenting in part b $x''(t) = \frac{t}{\sqrt{1+t^2}}$ and $y''(t) = \frac{2t}{2+t^2}$, even if not evaluated at $t = 4$, earned one of the last two points. But a number of such presentations contained chain rule errors. Similarly, in part d, students attempting $\int \ln(2+t^2)\,dt$ by hand had great difficulties, whereas the work that needed to be shown was the setup for the final answer, which merely required use of a calculator. This setup did not require that the explicit expression $\ln(2+t^2)$ be shown since $\ln(2+t^2)$ had been named, and certainly did not require display of an explicit expression for an antiderivative of $\ln(2+t^2)$.

(4) Part d asked for the total distance traveled along the curve over the time interval $4 \le t \le 6$. This is the integral of the speed which is the magnitude of the velocity vector. We have $|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

(5) Some students attempted part d using an arc length formula $\int_4^6 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dt$. Such students did not notice this should involve an integration with respect to x, $\frac{dy}{dx}$ in terms of x and limits on the integral of the x-coordinates of the particle at times $t = 4$ and $t = 6$. This approach was not handled successfully. Some other students tried a Pythagorean approach by using $\sqrt{\left(\int_4^6 \left(\frac{dx}{dt}\right) dt\right)^2 + \left(\int_4^6 \left(\frac{dy}{dt}\right) dt\right)^2}$. This approach could

occasionally be difficult for readers to catch since the result matches the correct answer to two decimal places after the point.