2022 AB/BC-4 — Chuck Garner, Rockdale Magnet School

Problem Overview

The student was given a table of selected values of the function $r'(t)$, where $r(t)$ is given as a twice-differentiable function. The function $r(t)$ models the radius of the base of a conical ice sculpture, which is melting.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

Part a

Students were asked to estimate $r''(8.5)$ by calculating the average rate of change over the interval $[7, 10]$, and to provide units of measure.

Part b

Students were asked to justify whether there is a time t in the interval $[0, 3]$ such that $r'(t) = -6$.

Part c

Students were asked to use a right-hand Riemann sum to estimate $\int_0^{12} r'(t)\, dt$.

Part d

Students were told that the height of the cone decreases at 2 cm/day and that at time $t = 3$, the radius is 100 cm and the height is 50 cm. Students were then asked to find the rate of change of the volume of the cone with respect to time, in cm^3/day, at $t = 3$. (Students were provided $V = \frac{1}{3}\pi r^2 h$.)

Comments on Student Responses and Scoring Guidelines

Part a

This part of the question could earn the student two points. The first point (the “answer point”) was earned if the student set up and evaluated a difference quotient using $r'(7)$ and $r'(10)$:

$$r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (-4.4)}{10 - 7}.$$

The student could stop at this stage and declare this as the answer to earn the point. The simplified answer is $0.6/3 = 0.2$.

To earn the answer point, the student had to show supporting work; a (correct) answer of 0.2 with no difference quotient did not earn the point. Any unsimplified numerical value must be equal to the correct answer to earn the point. Incorrect answers, or difference quotients that did not pull the values of $r'(7)$ and $r'(10)$ from the table, or used values other than $r'(7)$ and $r'(10)$, did not earn the answer point, but the student was eligible for the second point.

Students had to provide correct units of measure to earn the second point (the “units point”). Being the rate of change with respect to days of $r'(t)$, which is itself in centimeters per day, the units of $r''(8.5)$ are centimeters per day per day. This could also be reported as cm/day^2, or as (cm/day)/day or as cm/day/day. Students were eligible for the units point without earning the answer point, as long as some value of $r''(8.5)$ was presented. Correct units that were not attached to some value of $r''(8.5)$ did not earn the units point.

Part b

This part of the question could earn the student two points. The first point (the “inequality point”) is earned by the student reporting that -6 is between $r'(0) = -6.1$ and $r'(3) = -5.0$. This could be demonstrated by the student writing $-6.1 < -6 < -5$ or $r'(0) < -6 < r'(3)$, or by writing the two inequalities $r'(0) < -6$ and $r'(3) > -6$. The student could also express this verbally by writing “-6 is between $r'(0)$ and $r'(3)$”.

However, the following phrases did not earn this point since none of them put -6 specifically between -6.1 and -5 before making the conclusion.

- $-6.1 < r'(t) < -5$
- $r'(t) = -6$ because $r'(t) = -6.1$ and $r'(t) = -5$.
- $r'(t)$ goes from -6.1 to -5.
- It is melting ice which is continuous, so r' hits all the values between -6.1 and -5.
- $r'(t) = -6.1$ and $r'(t) = -5$ so by IVT $r'(t) = -6$.

The second point (the “justification point”) was earned provided the student met the following requirements.

1. The student said that $r'(t)$ is continuous because $r'(t)$ is differentiable. This was an important requirement. The student could not simply write “$r'(t)$ is continuous”. Nor could the student say “$r'(t)$

is differentiable and continuous”. The student was required to establish the implication that “$r'(t)$ is differentiable, therefore $r'(t)$ is continuous”.

2. The student earned the inequality point.
3. The student must conclude there is a time such that $r'(t) = -6$. (Simply writing “yes” met this requirement.)
4. If the name of a theorem is present, it must be the Intermediate Value Theorem (although the student need not name any theorem).

Part c

Two points are available to the student in part (c). To earn the first point (the “Riemann sum point”), the student had to present the right-hand Riemann sum, and the factors must be pulled from the table. A (correct) response earning the point is $3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5)$. If at most one of these eight factors was incorrect, the student still earned this point (but was ineligible for the second point).

To earn the second point (the “answer point”), the student must have the correct answer of -51. However, this need not be simplified. A response consisting solely of $3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earned both points.

Students had to show the sum of products to earn the Riemann sum point; if any one of the products was not shown the student did not earn this point (but was eligible for the answer point). For instance, responses such as $-15 + 4(-4.4) + 3(-3.8) - 7$ or $-15 - 17.6 - 11.4 - 7$ with no accompanying work did not earn the first point, but did earn the second since these evaluate to the correct answer. A reponse of only -51 with no supporting work earned no points.

A completely correct left-hand Riemann sum with a correct answer earned one point. If any mistake was present in the sum or the evaluation, it earned no points. Units were not required for this part and were ignored if provided.

Part d

In the last part of this problem, three points were available. The first two points were for using the product and chain rules correctly (the “product rule point” and the “chain rule point”) to obtain the derivative of V:

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right).$$

There were many responses which earned one of the two points for the derivative. The following are some of those responses.

$$\frac{dV}{dt} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2 \qquad \text{earns the product rule point,}$$

$$\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} \qquad \text{earns the chain rule point,}$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} \qquad \text{earns the chain rule point,}$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}h + \frac{dh}{dt}\right) \qquad \text{earns the product rule point.}$$

If a student wrongly assumed a proportional relationship between r and h and used it to eliminate either r or h the student could still earn the chain rule point. For instance, if the student assumed $h = 2r$, the student should get

$$V = \frac{1}{3}\pi(2h)^2h = \frac{4}{3}\pi h^3 \qquad \text{so that} \qquad \frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}.$$

That earns the chain rule point.

To earn the third point (the "answer point") the student must have our answer, which could be left either unsimplified as

$$\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100^2)(-2)$$

or simplified as $-\frac{70,000}{3}\pi$. In fact, a student response consisting only of the unsimplified answer shown above earned all 3 points for this part. Any mistakes in handling the constant $\frac{1}{3}\pi$ did not earn the answer point, but this did not disqualify the student from earning the derivative points.

Again, units were not required for this part and were ignored if given.

A student response of

$$\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dh}{dt} \qquad \text{or} \qquad \frac{dV}{dt} = \frac{2}{3}\pi r\frac{dr}{dt}\frac{dh}{dt} \qquad \text{or} \qquad \frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}\frac{dh}{dt}$$

did not earn any points at all.

<u>Observations and Recommendations for Teachers</u>

(1) Students should stop doing arithmetic. Many correct difference quotients in part (a) did not earn the answer point because of bad arithmetic. This Reader saw many examples of students trying to divide 0.6 and 3 and getting many answers other than 0.2. Students also had problems with the fact that both $r'(7)$ and $r'(10)$ are negative and both are decimals. Instead of calculating 0.6 for the numerator, students calculated 1.2, 1.8, -0.6, -1.4, and 1.4. Some students wrote $-3.8 - 4.4$ as the numerator of the difference quotient. Teachers should reinforce with their students that leaving an unsimplified correct answer earns points. A student simply writing

$$\frac{-3.8 - -4.4}{10 - 7} \text{ cm/day}^2$$

earned both points in part (a). (Even though we would like to see parentheses around -4.4, this did not render the student ineligible to earn the points.)

(2) Students should consider their answers. In part (a), students reported the units as cm^2/day, as cm/day^3, as cm/day, or as cm^3/day. Some of these incorrect units can be explained by poor arithmetic:

$$\frac{\text{cm/day}}{\text{day}} \qquad \text{was “simplified” as} \qquad \frac{\text{cm}}{\text{day}} \cdot \frac{\text{day}}{1} = \text{cm}.$$

Other incorrect units are simply the result of improper reasoning or not reading the problem carefully.

(3) Students should communicate well. A large number of students did not earn any points in part (b) due to lack of communication. Students seemed to have the right ideas, but did not state their ideas clearly. For example, a response such as

> “Since $r'(0) = -6.1$ and $r'(3) = -5$ and $r'(t)$ is continuous, there must be a time t where $r'(t) = -6$ by the IVT.”

did not earn either point available in part (b). A response such as this one did not put -6 between $r'(0)$ and $r'(3)$ nor was the differentiablility of r' given as the reason for it being continuous. Although the response seems to indicate that the student was thinking along the right lines, not enough of their reasoning was made clear by the student.

Poor communication was also a factor in part (c). Some students (more than this Reader could believe) set up the Riemann sum like this:

$$\int_0^{12} 3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5)\, dt.$$

While the Riemann sum itself is correct and evaluates to the correct answer, the presence of $3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ as an integrand renders the student ineligible for the Riemann sum point, but we still read for the answer point.

(4) Students should know the theorems they cite. Theorems are never required to be cited by name, but if they are, they should be correct. In part (b), students should have used the Intermediate Value Theorem to justify their reasoning. A large number of students cited the Mean Value Theorem even though they used the Intermediate Value Theorem. A few cited the Extreme Value Theorem, a few cited the Squeeze Theorem, and one cited L’Hôspital’s rule. Students should be instructed to never write the name of the theorem (and certainly never write it if they are unsure what the name is). Perfect work in part (b) was marred by the citation of the incorrect theorem. (Regurgitating the statement of the theorem without applying it to our problem never earns any points.)

(5) Students should stop doing arithmetic. Absolutely perfect Riemann sums in part (c) only earned one point due to errors in the arithmetic students felt they had to perform. Absolutely perfect derivatives in part (d), with the correct values substituted in the correct places, did not earn the answer point due to arithmetic errors in simplifying the numerical value. Excellent calculus was done by many students who should have earned 7, 8, or 9 points but only earned 4, 5, or 6 points due to arithmetic errors in parts (a), (c), and (d).

(6) Students should attend to precision. Four of the eight values for the Riemann sum in part (c) are negative. Many students dropped one or more of these negatives in writing or simplifying the answer. About 10% of papers encountered by this Reader had the correct Riemann sum, but reported an answer of positive 51. In part (d), many students did not earn a point for what would have been a correct answer due to missing parentheses. For their derivative in part (d), some students wrote

$$\frac{dV}{dt} = \frac{1}{3}\pi r\frac{dr}{dt}h + r^2\frac{dh}{dt} \qquad \text{or} \qquad \frac{dV}{dt} = \frac{1}{3}\pi \Big(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\ .$$

While both of these responses earned both derivative points, these mistakes in notation were carried over to the evaluation of the derivative, resulting in incorrect answers.

(7) Students should know how to handle problems in which a table of values is given. Some students believe that every Riemann sum problem is on equal subintervals. This was seen repeatedly in part (c) with the response $3(-5)+3(-4.4)+3(-3.8)+3(-3.5)$. The student has calculated $12/4 = 3$ as the width of each subinterval, even though the table clearly indicates this is not true. Worse, since there are now *two* errors in the Riemann sum, the student earns neither the Riemann sum point nor the answer point. Teachers should provide opportunities for students to handle problems where the function is given in a table with unequal subintervals throughout the entire school year.