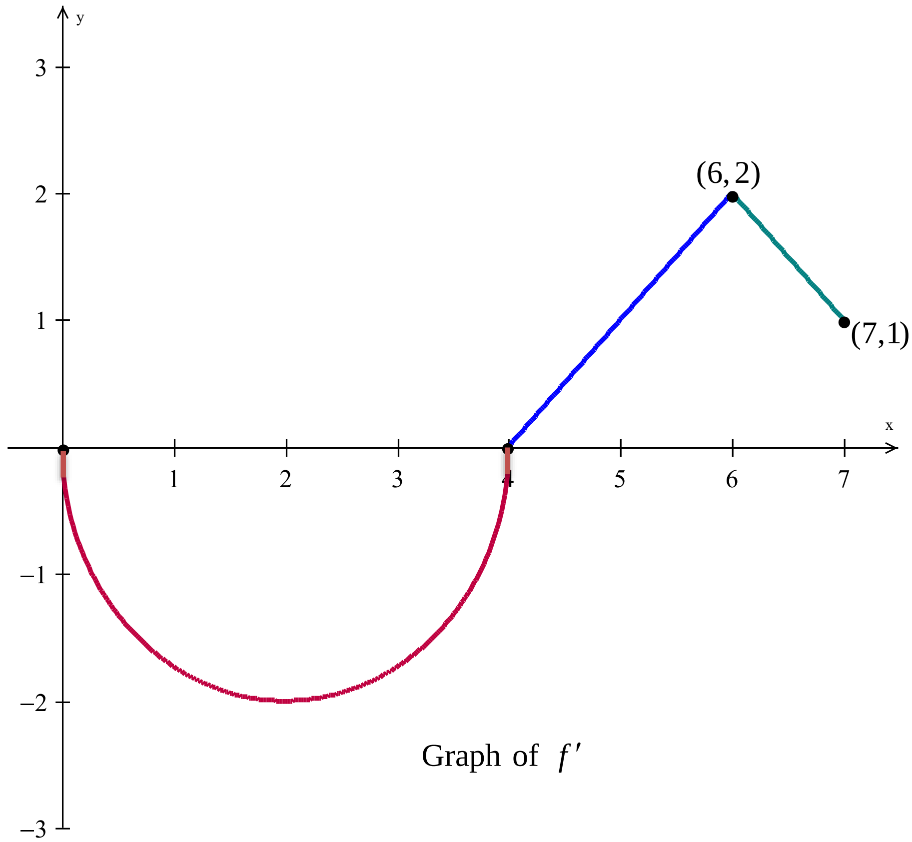
•

Problem Overview:



Graph of f'

f is given as a differentiable function with $f(4)=3$. On the interval $0 \le x \le 7$ the graph of f', the derivative of f, consists of a semi-circle and two line segments as shown in the figure above.

Part a:

Students were asked to find $f(0)$ and $f(5)$.

Part b:

Students were asked to find the *x*-coordinates of all points of inflection of the graph of f for $0 \le x \le 7$ and to justify the answer.

Part c:

The function g was defined by $g(x) = f(x) - x$. Students were asked to find the intervals, if any, on which g is decreasing for $0 \le x \le 7$ and to show the analysis that leads to the answer.

Part d:

Students were asked to find the absolute minimum value of g on $0 \le x \le 7$ and to justify the answer.

Comments on student responses and scoring guidelines:

Part a worth 3 points

The first point was earned for any of these:

(i) Finding the area of the semicircle
(ii) Finding the area of the triangle
(iii) Reporting the definite integral $\int_0^4 f'(x)\,dx$
(iv) Reporting the definite integral $\int_4^5 f'(x)\,dx$

The second and third points are for the values of $f(0)$ and $f(5)$. However, areas are easy to compute. Not much work needed to be shown. Thus:

(i) A response displaying $f(5) = \frac{7}{2}$ (with either a missing or incorrect value of $f(0)$) earned two of the three points in this part. In other words, since we have $f(5) = f(4) + Area\big|_{4 \le x \le 5}$, there is not much work required to be shown.

(ii) A response showing either $f(0) = \pm 2\pi$ or $f(5) = \frac{1}{2}$ or both earned one of the three points, even if no other correct work was shown.

A single value presented needed to be labeled as either $f(0)$ or $f(5)$. Two unlabeled values were read from left to right or top to bottom as $f(0)$ and $f(5)$.

Part b: worth 2 points

The first point was earned for the correct values of x at both inflection points. A response indicating any values other than 2 or 6 earned neither of the two points. In order to earn the second point, correct justification of these values required a reference to the graph of f' such as the change from increasing to decreasing at 6 and decreasing to increasing at 2 or a correct reference to the signs of the slopes of the graph of f'. A response referencing f'' is merely a statement of fact about an inflection point and did not earn the point for justification unless f'' was properly connected to the slopes of the graph of f'. Citing $x = 2$ and $x = 6$ as locations of extrema of f' was also an example of correct reasoning. A response indicating the inflection points as ordered pairs had to show correct values of the second coordinates, unless additional language as described above was present. These points are $(2,\ 3+\pi)$ and $(6,\ 5)$.

Part c: worth 2 points

Showing where $g(x) = f(x) - x$ is decreasing requires work with g'. Reporting that $g'(x) = f'(x) - 1$ earned the first point as did $f'(x) \le 1$ (no reference to *g* required) or something equivalent. However, the mere presence of $f'(x) - 1$ did not earn this point, there being no connection to either g' or 0. The second point was earned for specifying the interval $0 \le x \le 5$ (with or without endpoints) and the reason for this, which is that on this interval $g'(x) \le 0$. (For more information about endpoints, see "[Brackets] or (Parentheses)" by Chuck Garner in the GAAPMT May 2021 Newsletter posted at gaapmt.org.)

Part d: worth 2 points

The absolute minimum value of $g(x)$ occurs at $x = 5$ and is $-\frac{3}{2}$. Reporting that the sign of g' changes from negative to positive at $x = 5$ is a local argument and did not earn the second point for the justification and answer unless $x = 5$ was identified as the *only* critical point on the interval $0 \le x \le 7$. Alternatively, if the sign change at $x = 5$ was specified as $g' < 0$ for $0 < x < 5$ and $g' > 0$ for $5 < x < 7$, this global argument earned the answer/justification point. A response that assumed $g' = f'$ (an error imported from part c) should find the only critical point at $x = 4$, and if this was done, could earn the first point for the consideration of $g'(x) = 0$ somewhere in student work, but the second point could not be earned.

The search for an absolute extremum on a closed interval can be done using the candidates test, showing the values of the function at the endpoints of the interval and at any critical points. Students using this method were more successful on this part of the question than those trying to explain the sign change in g'. All three values of $g(x)$ at $x = 0$, 5 and 7 had to be correct. An incorrect value for $f(0)$ imported from part a could not be less than $-\frac{3}{2}$. A common error was assuming that $f(0) = 3 - 2\pi$. Using values less than $-\frac{3}{2}$. would contradict the conclusion from part c, which notes that $g(x)$ is decreasing on (0, 5).

Observations and recommendations for teachers:

(1) Many students knew to use areas in trying to find values of $f(x)$ in part a. Fewer seemed able to deal appropriately with the given value of $f(4)$ and sign issues. Thus, a common error was $f(0) = 3 - 2\pi$ which could impact work in part d. An integral definition of f was not given, but students should know this relationship between f and f'. A good starting point is to compute $\int_0^4 f'(x)\,dx = f(4) - f(0)$ which leads directly to the value $f(0)$. The only sign issue is that the area of the semi-circle is 2π, but the value of the definite integral $\int_0^4 f'(x)\,dx$ is -2π. Early work with definite integrals of elementary, easily graphed, functions should emphasize that the definite integral is not the sum of areas; rather, it is the sum of *signed* areas. In this early work with definite integrals, the definite integral is not area, but it does *relate* to areas.

(2) Numerical answers should be presented "labeled" as in $f(0) = n$ or $g'(1) = 6$ or $\left.\dfrac{dg}{dx}\right|_{x=1} = 6$. Merely stating numerical values is considered to be unsupported work. Often this earns no points on the exam.

(3) Regarding points of inflection (POIs) in part b, the general idea is to locate those at the values of x where concavity changes. Concavity can be related to f''. Given a graph of f', this second derivative is the slopes to that graph. If those slopes change signs, then a POI occurs. A not uncommon error was for a response to include the point where $x = 4$. This is a *possible* POI since POIs can be *possibly* found where $f' = 0$ or f' does not exist; and indeed f'' does not exist at $x = 4$ But the verification that a POI has been located is a change in the signs of f'' (slopes of f' or a change in the increasing/decreasing of f'). This type of analysis from a graph of f' should be practiced so that students know that a location where $f'' = 0$ or f'' does not exist is not automatically a POI. Also, students should be experienced with the idea that stating "f'' changes signs" requires that f'' be connected to slopes of f' to complete the analysis.

(4) The search for a reason for whether a function is increasing or decreasing (part c) should send students immediately to looking at the sign of the derivative. In part c, that requires showing some kind of work indicating consideration of $\dfrac{d}{dx}\left(f(x) - x\right) = f'(x) - 1$ and 0. Such work is usually worth one point on the exam. A look at the graph of f' shifted down 1 should show the obvious interval on which $g(x) = f(x) - x$ is decreasing. Students had difficulty putting that into words or remembering to provide the actual answer to the question, indicating a wonderful opportunity for problems which should be practiced.

(5) Finding an absolute extremum on a closed interval is often easiest to do using the candidates test. Students earning points in part d most often used this method. Because of the need to search for any critical points, correct work toward finding the location of critical points often earns a point on the test.

(6) Justifying that an absolute extremum occurs at a critical point on a closed interval is often difficult for students to do successfully by referencing changes in the sign of the first derivative. This is because this argument must properly analyze the signs throughout the entire interval (globally). In fact, most students working in part d used the testing of function values at the endpoints and critical point. Those who did not often left the argument incomplete by arguing only locally regarding a change in signs. Students should understand why a global argument is necessary and how to make that argument. However, perhaps more importantly, students should know how to examine values of the function at endpoints and any critical points when analyzing a function continuous on a closed interval.

(7) A reminder: Using the candidates test is applying the Extreme Value Theorem. This theorem is valid on a closed interval for a function continuous on that interval. Up through the 2022 exam it has not been required that students state or verify continuity. That could change. It is a good idea to note that continuity when applying the candidates test.

(8) Function analysis questions such as this one appear routinely on the AP Calculus Exam. For example, see 2015AB5, and AB/BC3 on the 2017, 2018 and 2019 exams.