2022 AB6 Marshall Ransom, Georgia Southern University

Problem Overview:

Two particles are in motion. Particle *P* moves along the *x*-axis, and its position is given by $x_P(t) = 6 - 4e^{-t}$ for $t > 0$. Particle *Q* moves along the *y*-axis, and its velocity is given by $v_Q(t) = \frac{1}{t^2}$ for $t > 0$. It is known that at time $t = 1$ the position of particle *Q* is $y_Q(1) = 2$.

Part a:

Students were asked to find $v_P(t)$, the velocity of particle *P* at time t.

Part b:

Students were asked to find $a_Q(t)$, the acceleration of particle *Q* at time t. Students were also asked to find all times t for $t > 0$ when the speed of particle Q is decreasing and to justify this answer.

Part c:

Students needed to find $y_Q(t)$, the position of particle *Q* at time t.

Part d:

Students needed to determine, as $t \to \infty$, which particle is eventually farther from the origin and to give a reason for the answer.

Comments on student responses and scoring guidelines:

Part a worth 1 point

Since velocity is the derivative of the position function, the answer is $\frac{d}{dt}x_P(t)=\frac{d}{dt}\left(6-4e^{-t}\right)=4e^{-t}$. No work needed to be shown. Some students had an issue with this by moving on to substitute 1 for t. These responses did not earn this point. A few errors in differentiation involved trying to use the power rule.

Part b: worth 3 points

The expression $\frac{-2}{t^3}$ earned the first point since $a_Q(t)=v'_Q(t)$. Earning this first point was not a necessary eligibility for earning the second and third points. But an expression for $a_Q(t)$ was necessary in order to earn the third point. The second point was earned for considering signs of both $a_Q(t)$ and $v_Q(t)$, whether the sign for $a_Q(t)$ was correct or not. A response had to earn the second point in order to be eligible for the third. Thus, an erroneous expression in part a could lead to something like "signs of $a_Q(t)$ and $v_Q(t)$ are positive, so no time at which decreasing" which earned the last two points if the presented $a_Q(t)>0$. However, a response appealing to $v_Q(t)<0$ was not eligible for the third point.

Part c: worth 3 points

One solution involves adding the initial position to the displacement of the particle. Since the initial position is given for $t=1$, the definite integral required is the integral of $v_Q(t)$ from 1 to t. Use of this integral earned the first point. If the integral was presented improperly and misused variables as in $\int_1^t \frac{1}{t^2}\,dt$, the first point for the integral was not earned unless there was an attempt at computation of an integral. A response presenting either $\int \frac{1}{t^2}\,dt$ or $-\frac{1}{t}$ earned this first point. The second point was earned for using the initial condition. And so, if $\int \frac{1}{t^2}\,dt$ was presented and was followed by $2=-1+C$, the second point was also earned. The third point was for the answer. The antiderivative and the value of C were easy to compute, and the response $-\frac{1}{t}+3$ earned all three points while $-\frac{1}{t}+C$ with no additional work earned only the first point. A second method using an indefinite integral followed by solving for C was seen more often than use of the definite integral.

Part d: worth 2 points

The first point was earned for at least one of the limits $\lim_{t\to\infty}\left(6-4e^{-t}\right)=6$ or $\lim_{t\to\infty}\left(3-\frac{1}{t}\right)=3$. Language such as " $x_P(t)$ goes to 6" or " $y_Q(t)$ approaches 3" earned this point. A response with an incorrect $y_Q(t)$ from part c was eligible for both points provided that $y_Q(t)$ was a non-constant function. The second point was for using both limits and providing an answer consistent with the presented $y_Q(t)$. In the case of working the problem correctly, since $6>3$, particle P is eventually farther from the origin.

Observations and recommendations for teachers:

(1) It is disappointing to see responses trying to use the power rule to calculate $\frac{d}{dt}\left(6-4e^{-t}\right)$. Once the background has been established showing that $\frac{d}{dt}e^{t}=e^{t}$, it can be emphasized that the first step in calculating the derivative of $e^{p(x)}$ is simply "copy it down." Apply the chain rule, multiplying by $\frac{d}{dt}p(x)$. That being said, it was clear that most students trying to use the power rule were lacking in other knowledge as well.

(2) Parts a, b and c show once again that computation of derivatives (as well as antiderivatives) is an important skill required on the AP Calculus Exam.

(3) Particle motion on an axis speeds up or slows down depending upon the relative signs of the acceleration and velocity. If signs are the same, the velocity and acceleration are in the same direction, and speed is increasing; and if the signs are opposite, speed is decreasing. Same and opposite signs can also be indicated by showing the product of acceleration as either positive or negative respectively.

(4) Part c was approached by students using one of two methods:

(i) One required use of the evaluation part of the Fundamental Theorem. That is, if $A(x)$ is an antiderivative of $f(x)$ and $f(x)$ is continuous on [a, b], then $\int_a^b f(x)\,dx=A(b)-A(a)$. The useful form of this evaluation for part c is $A(a)+\int_a^b f(x)\,dx=A(b)$ where $A(x)$ is the position function, $f(x)$ is the velocity function and $A(a)$ is the "initial position" of the particle in motion for $t=a$.

(ii) One method is to compute the indefinite integral and use the initial condition to solve for C. More students seemed to use this method and to be successful in solving the problem. In this case, we have $y_Q(t) = \int \frac{1}{t^2}\,dt = -\frac{1}{t} + C$. Using the initial condition, we have $2 = -1 + C$ leading to $y_Q(t) = -\frac{1}{t} + 3$.

(5) A number of students used ∞ in part d as though it were a numerical value as in $\lim_{t\to\infty}\left(3 - \frac{1}{t}\right) = 3 - \frac{1}{\infty} = 3$. Such responses did not earn the first point in part d. While it is all right to "think" about the forms $\frac{1}{\infty} = 0$ or $e^{-\infty} = \frac{1}{e^{\infty}} = 0$, **<u>these forms do not make sense numerically and should not be written</u>**, especially with equal signs. If written as part of the response, this must clearly be shown as scratch work only, or better yet crossed out.

(6) A third method for solving part c involves using two definite integrals to solve the separated differential equation $dy_Q(t) = \frac{1}{t^2}\,dt$ subject to the condition that $y_Q(1) = 2$. The initial condition is used as the two lower limits on the integrals. Using f to represent the desired function and s as a "dummy variable," we have $\frac{dy_Q}{dt} = \frac{1}{t^2}$ and $y_Q(1) = 2 \rightarrow \int_2^f dy_Q = \int_1^t \frac{1}{s^2}\,ds \rightarrow f - 2 = -\frac{1}{t} + 1$. The desired solution is $f = -\frac{1}{t} + 3$. I have only very rarely seen this method used on the AP Calculus Exam for solving a separable differential equation, given an initial condition. But it works! All that's happening is that the initial condition is being used as the definite integrals are evaluated.