2022 AB-5 Chuck Garner, Rockdale Magnet School

Problem Overview

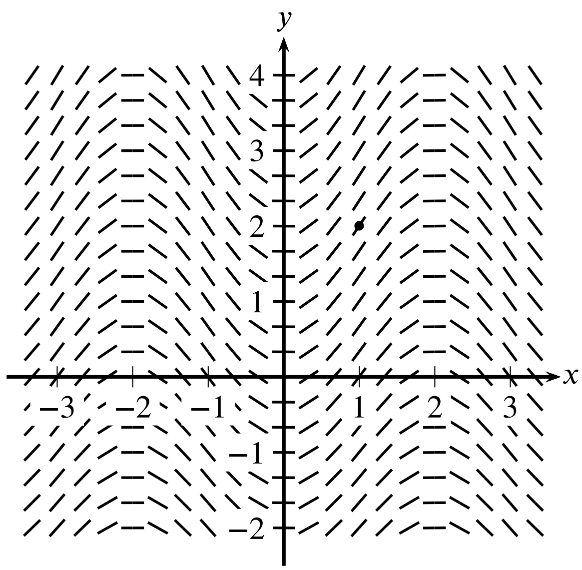
The student is given the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

with the initial condition that $f(1) = 2$, where $y = f(x)$ is the particular solution. The student is told that f is defined over all real numbers.

Part a

Students were given the slope field below and asked to sketch the solution curve through $(1, 2)$.



Part b

Students were asked to write the equation of the tangent line to the solution curve at $(1, 2)$ and then to use it to approximate $f(0.8)$.

Part c

Students were asked to determine whether the approximation in part (b) is an overestimate or underestimate, given that $f''(x) > 0$ for $-1 \leq x \leq 1$. Students were also asked to give a reason for their answer.

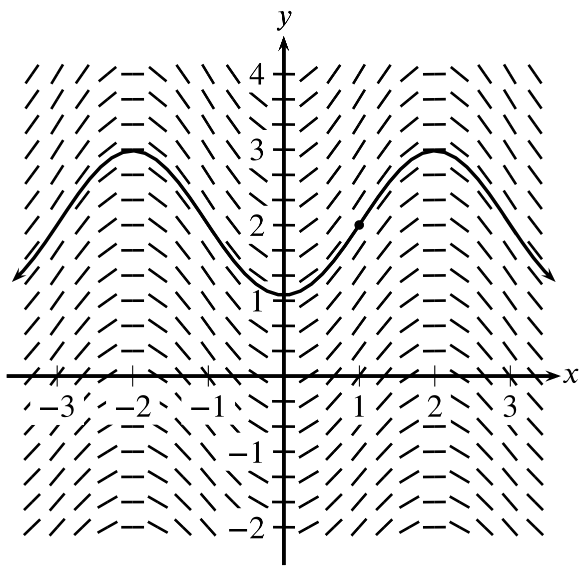
Part d

Students were asked to find the particular solution of the differential equation, with the initial condition $f(1) = 2$, using separation of variables.

Comments on Student Responses and Scoring Guidelines

Part a

This part could earn the student one point. The solution curve is shown below.



To earn the point, the student's solution curve had to satisfy the following requirements.

1. The curve must pass through the point $(1, 2)$.
2. The curve should extend "reasonably close" to the edges of the slope field.
3. The curve cannot have conflicts with the given slopes.
4. The values of $f(x)$ must be positive for all points on the curve.
5. Any extrema must occur at horizontal slopes.

If the student's curve extended past the slope field, those portions of the curve were ignored.

Part b

This part of the question could earn the student two points. The first point (the "tangent line point") is earned by the student reporting the tangent line as $y = \frac{3}{2}(x - 1) + 2$, or in some equivalent form. Leaving the slope as $\frac{3}{2}\sin(\frac{\pi}{2})$ is accepted (if not ideal). If the student reports an incorrect tangent line, but still uses a correct slope of $\frac{3}{2}$, the student does not earn the tangent line point but is eligible for the second point. If the student does not calculate the slope at $(1, 2)$ correctly, the student does not earn either point.

The second point (the "approximation point") was earned if the student provided the approximation $\frac{3}{2}(0.8 - 1) + 2 = 1.7$. If the student's only response was $\frac{3}{2}(0.8 - 1) + 2$, the student earned the approximation point, but not the tangent line point as the student did not provide an equation of a line.

Part c

One point is available to the student in part (c). To earn the point, the student must report that the approximation is an underestimate and must provide a reason. The acceptable reasons the student could provide include the given information that $f''(x) > 0$, that $f'(x)$ is increasing, or that $f(x)$ is concave up. A response that included something like "the graph is positive," "the graph is concave up," "the derivative is increasing" were not accepted due to ambiguous reference to "the graph" or "the derivative."

Part d

In the last part of this problem, five points were available. The first point was awarded for separating variables correctly into

$$\frac{dy}{\sqrt{y+7}} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx,$$

with or without integral signs. A response with no separation of variables earns none of the five points.

The second point was earned with one correct antiderivative; the third point was earned with a second correct antiderivative. The student's antiderivatives should be

$$2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right) + C.$$

The fourth point was earned if the student included the constant of integration *and* found the initial value using the initial condition. Using $f(1) = 2$, we have

$$2\sqrt{2+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\cdot 1\right) + C$$

so that

$$6 = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) + C,$$

and we conclude that $C = 6$. To be eligible for this point, the student must

- correctly include the constant of integration in an equation;
- substitute 1 for x and 2 for y;
- have earned the first point;
- have earned one of the two antiderivartive points. (In other words, the student with two incorrect antiderivatives can at most earn the first point.)

A response with no constant of integration can, at most, earn the first three points.

The fifth and final point is earned by solving for y. From

$$2\sqrt{y+7} = 6 - \frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)$$

we have

$$y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7.$$

The student was only eligible for the fifth point if the student earned the first four points.

There was a special allowance made for students who separated incorrectly by writing

$$\sqrt{y+7}dy = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx.$$

This response did not earn the first point, was eligible for the antiderivative point in terms of x, and was eligible for the fourth point. This reponse did not earn the fifth point.

Observations and Recommendations for Teachers

(1) Sketching a solution curve thorugh a slope field is a required skill in the AB course. Whereas this was a relatively easy point to earn, it was also an easy point to slip through students' fingers. Teachers should allow students some practice with this skill before the AP Exam.

(2) Students should consider their answers. In part (b), some students had the correct reason ($f''(x) > 0$ or $f(x)$ is concave up), but responded with the opposite answer of an overestimate. Whether this was due to conceptual misunderstanding or lack of attention to detail is unknown. In either case, such a response did not earn the only point available.

(3) Students should stop doing arithmetic. A good number of students did not earn both points in part (b) due to arithmetic errors in simplifying $\frac{3}{2}(0.8-1)+2$. Some also indicated the wrong value of $\sin(\frac{\pi}{2})$ (this mistake rendered the students ineligible for both points).

(4) Students should stop thinking that the antiderivative of every fraction involves the natural logarithm. The mistaken belief that an antiderivative of $1/\sqrt{y+7}$ is $\ln\sqrt{y+7}$ knocked many students out of earning points. Consider that this mistake paired with a mistake in the antiderivative of the sine term (such as an incorrect coefficient in the antiderivative), meant that the student could only earn the first point.

(5) Part (d) has become the standard expectation for students when asked to solve a differential equation, and has become the standard point breakdown. (However, sometimes the antiderivatives are combined into one point, making the total points four rather than five.) Teachers should give students differential equations that require not only separation of variables, but that also involve two not necessarily straightforward antiderivatives.