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This publication includes:

Note from the President

When I think back to teachers and professors that most impacted my life, the common denominators were their passion for their content area and their passion for teaching. I can tell that you share these passions and so impact your students' lives. You are passionate about calculus or statistics, and you are passionate about teaching. Otherwise you wouldn't be reading this newsletter, working to hone your craft, and attempting to always find new ways to reach your students. I applaud you and on behalf of Georgia students, parents, educators, and other stakeholders thank you for your continued effort to inspire students, even through such a challenging school year.

The GA^{2}PMT invites you this year to join us for professional development in several ways. First, we would love to see you online at the annual Georgia Math Conference in October. We are planning several AP Statistics and AP Calculus sessions that I think you will really enjoy. For example, one of our AP Calculus sessions will discuss modeling COVId-19 Data with Differential Equations. We will also have reports from this year's AP Math readings with teachers who have scored tests over the summer. Second, we are working on the possibility of offering an in person mini-conference in the near future. We are monitoring COVID and the CDC recommendations for in person meetings and will send you more information about the mini-conference once we feel that holding such an event will be safe for our members. Additionally, COVID has forced us to pause our plans to sponsor an AP Calculus tournament but we are hoping to finally hold the event in 2022. More details about the tournament will be shared once the plans are finalized.

Maybe next year we will be able to return to Rock Eagle and visit with each other in person. We've certainly missed seeing you all. One thing that I've learned from watching you and the other teachers across our state is that you are meeting the challenges of this school year head on and doing the best that you can to prepare your students. You are an inspiration to your students and you are an inspiration to me. We hope to see you online at the Georgia Math Conference this year!

Sincerely

Billy W. Esra II

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The Green Monster

Adapted from Calculus Projects Using *Mathematica*, Georgia Institute of Technology

Lina Ellis
The Westminster Schools

The left field wall at Fenway Park is known as the Green Monster because it is particularly hard to hit a ball over it and it is green! It is 315 feet from home plate and 37 feet high. Let's model the path followed by a batted ball assuming there is no air resistance (a big assumption!).

1. Write a set of parametric equations to model the path of the ball if it is initially hit from a height of *h* feet with a launch angle of *a* and an initial velocity of *v*.
2. Assuming the ball is hit towards the Green Monster from 3 feet above home plate with an initial velocity of 180 feet per second, graph the path of the ball using a launch angle of your choosing.
3. Does the player make a home run? Support your answer both graphically and algebraically.
4. Experiment to see which launch angles will yield a home run for an initial velocity of 180 ft./sec.
5. Now, algebraically find the range of angle values that will yield a home run. You can use technology once you have the equations set up.
6. Experiment to see what happens when the initial velocity is smaller. What angles now produce a home run?
7. Determine the minimum initial speed that produces a home run. Justify your answer. You may use technology once you have set up the problem.
8. Prove that the range of the ball (the horizontal distance) is maximized when a = 45 degrees.
9. Find the angle that maximizes the total distance travelled by the ball. You may use technology once you have set up the problem.
10. Multivariable Calculus Extension: Find the function f(a,v) that represents the height of the ball when it reaches the Green Monster as a function of the initial angle and the initial velocity of the ball.

The Green Monster Solutions

Adapted from Calculus Projects Using *Mathematica*, Georgia Institute of Technology

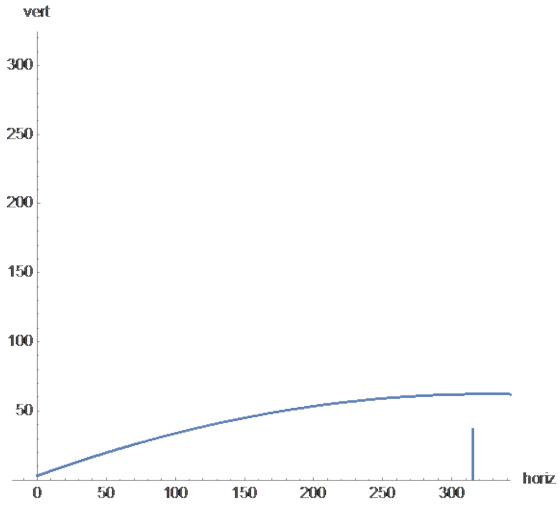
Lina Ellis
The Westminster Schools

The left field wall at Fenway Park is known as the Green Monster because it is particularly hard to hit a ball over it and it is green! It is 315 feet from home plate and 37 feet high. Let's model the path followed by a batted ball assuming there is no air resistance (a big assumption!).

1. Write a set of parametric equations to model the path of the ball if it is initially hit from a height of *h* feet with a launch angle of *a* and an initial velocity of *v*.

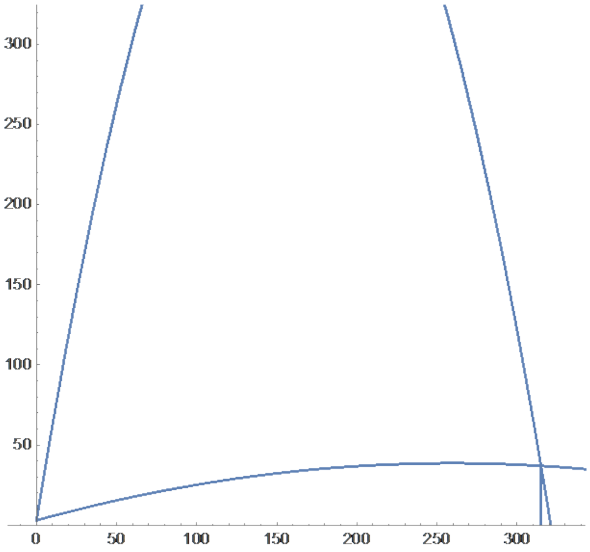
   $x = (v \cos a)t$
   $y = -16t^2 + (v \sin a)t + 3$

2. Assuming the ball is hit towards the Green Monster from 3 feet above home plate with an initial velocity of 180 feet per second, graph the path of the ball using a launch angle of your choosing. Assuming a = pi/9:



3. Does the player make a home run? Support your answer both graphically and algebraically. Yes. When (180 cos(pi/9) t = 315, t = 1.862 sec. At t = 1.862, y = 62.159 ft.

4. Experiment to see which launch angles will yield a home run for an initial velocity of 180 ft./sec.



5. Now, algebraically find the range of angle values that will yield a home run. You can use technology once you have the equations set up.
   15.3912 < a < 80.7693

6. Experiment to see what happens when the initial velocity is smaller. What angles now produce a home run?

7. Determine the minimum initial speed that produces a home run. Justify your answer. You may use technology once you have set up the problem.
   Solve[3 - (1587600 Sec[a]^2)/v^2 + 315 Tan[a] = 37, v], then use the first derivative test. The minimum speed occurs when a = .839158 radians at a speed of 105.955 ft per sec.

8. Prove that the range of the ball (the horizontal distance) is maximized when a = 45 degrees.
   Find t when y = 0. Substitute in to x. Use the first derivative test on x.

9. Find the angle that maximizes the total distance travelled by the ball. You may use technology once you have set up the problem.
   Use the first derivative test on the arclength formula. a = ~ 56 degrees.

10. Multivariable Calculus Extension: Find the function f(a,v) that represents the height of the ball when it reaches the Green Monster as a function of the initial angle and the initial velocity of the ball.
    f(a,v) = 3 - 1587600 Sec[a]^2)/v^2 + 315 Tan[a]

AP Statistics Resources

Vicki Greenberg
The Lovett School

There are some really great resources out there for teachers of AP Statistics!

The first one is Stats Medic by Luke Wilcox and Lindsey Gallas (https://www.statsmedic.com/). Their mission is to transform statistics education by empowering teachers to lead student-centered classrooms. Students and teachers all over the world are loving their resources. And, as if it could not get any better, check out this google sheet which lists Desmos activities that correspond to the lessons (Teaching Stats Medic Lessons Virtually)! There is even a Facebook group (Stats Medic Teacher Community).

The second one is Skew the Script by Dashiell Young-Saver (https://skewthescript.org/). Skew the Script offers free curriculum that makes math what it already is: relevant. What is the relationship between police use of force and race? What was Kobe’s probability of scoring 81 points in a game? What is the most effective way to prevent strokes? These are all questions our curriculum tackles through statistics, mathematics, and critical thinking. There are also videos that teaches the lessons and desmos activities for most lessons – perfect for distance learning! And, of course, there is a Facebook page (Skew The Script AP Stats Teachers' Lounge).

I also want to highlight Leigh Natario and Bob Peterson. They feature AP teachers from across the globe willing to share lessons, ideas, and activities that foster student engagement in both a virtual and traditional classroom setting in a webinar about twice a month on Monday nights (zoom link). Even if you cannot make it, they post the recordings on their youtube channel (Teaching Statistics Across Any Setting). Reach out to Leigh or Bob with questions: statsvirtual@gmail.com.

Finally, I want to highlight the AP Daily Videos that the College Board has produced. These are great for distance learning and for extra help when kids need it. These videos are located in AP Classroom. You can find more information here.

AP SUMMER INSTITUTES

https://eventreg.collegeboard.org/c/calendar/54ae034d-96ef-4609-8458-c7c7a76ad3b9

[Brackets] or (Parentheses)

Chuck Garner, Ph.D.
Rockdale Magnet School for Science and Technology

So here you are with your calculus class, teaching students how to excel on those wonderful problems that ask for extrema, critical points, and intervals of increasing and decreasing. Your lesson goes extremely well, and everyone has got it—congratulations! But then the students come into class the next day with their inevitable questions: "How do I know when to use brackets or parentheses on the intervals?" Another student says, "Yeah, on the very first problem, the one with $y = x^3 - 12x$, the interval where this is decreasing is from $x = -2$ to $x = 2$, but is the answer $(-2, 2)$ or is it $[-2, 2]$?" And another students pipes up with the common refrain "What is accepted on the AP Exam?" And that sparks some debate and more questions, and frankly, you're not sure yourself—about any of that!

Does that sound familiar to you? If so, that's pure speculation on my part that it does, because what I described was my class, about 10 years ago! The short answer is that the AP Exam Readers are not picky about brackets or parentheses for functions defined over all real numbers—there could be some execeptions based on domain issues, but in general, either are acceptable. However it brought to my mind the question concerning the actual use of brackets or parentheses: which is mathematically correct? I've had discussions about this with other calculus instructors which interested me enough to seek out some answers to this question, which I'd like to share with you.

A Definition. The most interesting thing I learned in my initial discussions and investigations was that the definition of an increasing or decreasing function has nothing to do with calculus. Yes, that's right, we need no calculus to define an increasing function. The definition of an increasing function is below.

> A function $f(x)$ *increasing* on an interval I if and only if $f(b) \geq f(a)$ for all $b > a$, where $a, b \in I$. If $f(b) > f(a)$ for all $b > a$, then the function is *strictly increasing*.[1]

(There is a similar definition for a decreasing function, but we simplify our discussion by considering only increasing.) Three things about this definition jump out: equality is allowed; the lack of the condition of continuity; and there are no brackets or parentheses!

The possibility of equality implies that constant functions—their graphs are horizontal lines—are increasing functions. Of course, in our classes, we do not want to consider a function such as $f(x) = 9$ as increasing. So I arrived at the first thing I did incorrectly from a mathematical point of view: the notion of increasing functions that I taught my students was really the notion of *strictly* increasing. I had never said the phrase "strictly increasing" to my students. I simply said that a function like $f(x) = 9$ was neither increasing nor decreasing, which is true, depending on the

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[1] From (with a change in notation) page 178 of G.H. Hardy's *A Course of Pure Mathematics*, tenth edition, Cambridge, 1952. Hardy uses the term "steadily increasing" instead of "strictly increasing." This definition also appears on page 6 (!) of J. Rogawski, *Calculus: Early Transcendentals*, first edition, 2008 and page 19 of J. Stewart, *Calculus*, seventh edition, 2012.

definition you use. And that brought me to the fact that definitions matter! (This was a ridiculously slow realization on my part, since I am familiar with the deductive systems of geometries and abstract algebra, which always begin with axioms and definitions.) We should clearly define for our students the definition of increasing that we will be using in class. I personally chose to remove the possibility of equality from the definition of increasing that I use, for no other reason than to avoid constantly saying "strictly increasing" and we can just say "increasing."

The lack of continuity in the definition is an interesting aspect. Consider the function

$$g(x) = \begin{cases} x+1 & x < 0 \\ x+2 & x \geq 0. \end{cases}$$

This function is has a jump discontinuity at $x = 0$ and is therefore not differentiable at $x = 0$. However, by the definition, it is increasing on any interval.

As for the brackets versus parentheses question, consider this. The definition says that the property must apply for all $b > a$ where $a, b \in I$. Then consider the simple function $f(x) = x^3$ on the closed interval $[-1, 1]$. Take any point $b \in I$ such that $b > 0$. Then it is still true that $f(b) > f(0)$. Likewise, take any point $a \in I$ such that $a < 0$. Then it is also true that $f(0) > f(a)$. Then by the definition, this function is increasing on this interval, even though the derivative is zero at $x = 0$. But is it increasing on the closed interval or on the open interval? To see which, take any point $a \in I$ such that $a < 1$. Then we do indeed have $f(1) > f(a)$. Take any point $b \in I$ such that $b > -1$. Then we also have $f(b) > f(-1)$. Therefore, the function $f(x) = x^3$ is increasing on the *closed* interval $[-1, 1]$.

To be more precise in showing that $f(x) = x^3$ is increasing on this interval (indeed, on any interval), we should work with the inequality from the definition directly. Let $x = x_0$ and consider a point $x_0 + h$ where $h > 0$. Then we need to show that $f(x_0 + h) > f(x_0)$ for all x_0 and h. Note that since $h > 0$, then $x_0 + h > x_0$. Hence,

$$\begin{aligned} x_0 + h &> x_0 \\ (x_0 + h)^3 &> x_0^3 \\ x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 &> x_0^3 \\ 3x_0^2h + 3x_0h^2 + h^3 &> 0 \\ h(3x_0^2 + 3x_0h + h^2) &> 0 \\ h\left(3\left(x_0 + \frac{1}{2}h\right)^2 + \frac{1}{4}h^2\right) &> 0. \end{aligned}$$

This last inequality was obtained by completing the square, and the expression on the left-hand side is always positive for any x_0. Thus, $f(x) = x^3$ is increasing over any interval. Now this takes far too much time to do with the interesting functions we show students in calculus class, and I do not show them any of this. What we have instead is a short-cut using the derivative, and that is what they learn. However, this short-cut has a caveat, as many short-cuts do.

Some Calculus. When we use the first derivative test to determine intervals of increasing or decreasing, we use the following result. This result can be proved using the definition of increasing and the Mean Value Theorem, with the assumption that f is differentiable.

> Let $f(x)$ be a continuous function. If $f'(x) > 0$ on an open interval (a, b), then $f(x)$ is increasing on (a, b).

By this result,[2] we see that the derivative of $f(x) = x^3$ is positive on the intervals $(-1, 0)$ and $(0, 1)$, and is therefore increasing on these intervals. Doesn't this contradict the definition of an increasing function? No, it does not. The result above *does not imply* that if a function is increasing on an open interval, then the derivative is positive on that interval. In other words, the result above is not an "if-and-only-if" statement. It is possible for a function to be increasing on an interval and to also not be differentiable, like the piece-wise function g above. It is also possible for a function to be increasing on an interval while the derivative is zero at a finite number of points on that interval, like the function $f(x) = x^3$. Moreover, we can say that $f(x) = x^3$ is increasing on the closed interval $[-1, 1]$ because the closed interval meets the definition of an increasing function even though the result above applies only to open intervals.

An Example. Let's find the intervals where $h(x) = (x-1)^3(x-5)$ is increasing and intervals where $h(x)$ is decreasing. Following the first derivative test, we take the derivative and set it to zero:

$$\begin{aligned} h'(x) = 3(x-1)^2(x-5) + (x-1)^3 &= 0 \\ (x-1)^2(3(x-5) + x - 1) &= 0 \\ (x-1)^2(4x-16) &= 0 \\ 4(x-1)^2(x-4) &= 0 \\ x &= 1, 4. \end{aligned}$$

Since $h'(x) < 0$ for $x < 1$ and $1 < x < 4$, and $h'(x) > 0$ for $x > 4$, h is decreasing on $(-\infty, 1)$ and $(1, 4)$, and increasing on $(4, \infty)$. However, where does this leave $x = 1$ and $x = 4$? The case where the derivative is 0 is not covered by the first derivative test: the test says that *if* the derivative is positive (or negative), *then* we may conclude something about the function increasing (or decreasing). When the derivative is neither positive nor negative, *then we cannot use the first derivative test*, and we must appeal to the definition of increasing (or decreasing). This reasoning implies that h is decreasing on $(-\infty, 4]$ and increasing on $[4, \infty)$. Note that $x = 1$ can be legitimately placed within the interval $(-\infty, 4]$, and we can include $x = 4$ in both intervals.

This may seem like a horrible mistake to include $x = 4$ in both intervals! How could $x = 4$ be both increasing and decreasing? Well, it cannot, but not for the reason you may think. The definition of increasing (and decreasing) applies to *intervals*, not *points*. So it is incorrect to refer

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[2] The proof is by contradiction. Assume $f'(x) > 0$ and that f is not increasing on I. Then there are $a, b \in I$ such that $f(a) > f(b)$. By the Mean Value Theorem, $f'(c) = (f(b) - f(a))/(b - a)$ for some $c \in (a, b)$. Since $f(a) > f(b)$, then $f(b) - f(a) \leq 0$. Since $a < b$, then $b - a > 0$. Hence, $(f(b) - f(a))/(b - a) = f'(c) \leq 0$. But this contradicts the assumption that $f'(x) > 0$ on I. The result follows.

to a function increasing at a single point. Hence, this function does not increase or decrease *at the single point* $x = 4$, and it also does not increase or decrease at any other single point, either.[3]

With that aside, consider the definition. We do have that for any $b \in [4, \infty)$ where $b > 4$, we also have $h(b) > h(4) = -27$; so h is increasing on this interval. Likewise, for any $a \in (-\infty, 4]$ where $a < 4$, we also have $h(a) > h(4) = -27$; so h is decreasing on this interval. Thus, $x = 4$ can be correctly placed within both intervals and we can use brackets on that side of the interval.

That is so much to think about for the typical calculus student, and that is a good reason why the AP Exam Readers would accept either brackets or parentheses.

Concavity. If we know that the function in question has a second derivative, then a necessary and sufficient condition for concavity is determined by the second derivative. This gives us the "rule" we teach our calculus students:

> If $f''(x)$ exists and $f''(x) \geq 0$ in $[a, b]$, then $f(x)$ is concave up on $[a, b]$. If $f''(x)$ exists and $f''(x) \leq 0$ in $[a, b]$, then $f(x)$ is concave down on $[a, b]$.

Again, notice the explicit use of the closed interval. However, you may remember that with increasing and decreasing, we really use these terms to mean *strictly* increasing and *strictly* decreasing. We have the same situation here. What we traditionally discuss is that a function is concave up when $f''(x) > 0$, which means we are actually talking about a function being *strictly* concave up. We could make the decision to always mean "strictly concave up" when we say "concave up." Therefore, using this convention, we would not accept a function being concave up on $[7, \infty)$, but only on the interval $(7, \infty)$. So it seems when it comes to concavity, we can decide firmly the question of brackets or parentheses: we always use parentheses. Oh, but there's a twist: consider another definition of concavity.

> If $f'(x)$ exists and is increasing on an interval I, then f is concave up on I. If $f'(x)$ exists and is decreasing on an interval I, then f is concave down on I.[4]

This definition has some interesting implications as well: f must be differentiable to have concavity and the points of inflection are the endpoints of the increasing or decreasing intervals of f'. The second derivative only tells us concavity on the open interval, but this definition, which is related to the definitions of increasing and decreasing, implies concavity can occur on the closed

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[3] Yes, there are many questions in many books where questions are posed like "Is f increasing at $x = 5$?" In this case, we should just narrow our eyes, glance away, and understand that the question is really asking "Is f increasing on the interval $[5 - \epsilon, 5 + \epsilon]$ for some small $\epsilon > 0$?" Indeed, this is equivalent to E. Landau's definition of "increasing at a point" where the point in question is the center of an interval of width 2ϵ. This appears on page 88 in Landau's *Differential and Integral Calculus*, third edition, AMS-Chelsea Publishing, 1960.

[4] From page 207 of Finney, Demana, Waits, Kennedy, *Calculus: Graphical, Numerical, Algebraic*, third edition, Pearson, 2007. This definition also appears on page 239 of J. Rogawski, *Calculus: Early Transcendentals*, first edition, 2008.

interval. We will find the concavity of $h(x) = (x-1)^3(x-5)$.

$$\begin{aligned} h''(x) = 8(x-1)(x-4) + 4(x-1)^2 &= 0 \\ 4(x-1)(2(x-4) + x - 1) &= 0 \\ 4(x-1)(3x-9) &= 0 \\ 12(x-1)(x-3) &= 0 \\ x &= 1, 3. \end{aligned}$$

We find that $h''(x) > 0$ for $x < 1$, $h''(x) < 0$ for $1 < x < 3$, and $h''(x) > 0$ for $x > 3$. This implies that $h'(x)$ is increasing for $(-\infty, 1]$ and $[3, \infty)$ and therefore h is concave up on these intervals. We also have that $h'(x)$ is decreasing on $[1, 3]$, and therefore h is concave down there.

Once more, we have a seemingly incorrect statement that h is both concave up and concave down at $x = 1$ and $x = 3$. However, concavity, like increasing/decreasing, refers to intervals, not points. If we are to be consistent with our use of closed intervals for increasing/decreasing, then we should be consistent with closed intervals for concavity, if we use the definition of concavity based on an increasing/decrasing derivative. Definitions matter! As another example of this, consider another definition of concavity.

> If the graph of $f(x)$ lies above all of its tangents on an interval I, then it is concave up on I. If the graph of $f(x)$ lies below all of its tangents on I, it is concave downward on I.[5]

Are these intervals open or closed under this definition? Consider an endpoint of one of these intervals, and assume the function f has an inflection point at this endpoint. Then there is a tangent line at this endpoint, but this tangent cannot lie entirely above or below the function. Hence we would not include the endpoint in an interval describing concavity, and thus the interval is open. Under this definition, we would need to use parentheses every time we describe an interval of concavity.

Concavity without Calculus. But all of the preceeding statements of concavity rely on calculus. There is a definition of concavity which does not rely on calculus, just as there is one for increasing. That definition is below.

> A continuous function $f(x)$ is concave up on an interval I if, and only if, for any $a, b \in I$, and any t such that $0 < t < 1$, $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$. Likewise, f is concave down if, and only if, $f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$.[6]

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[5] From page 216 of J. Stewart, *Calculus*, seventh edition, 2012.

[6] From (with a change in notation) page 61 of W. Rudin, *Real and Complex Analysis*, third edition, 1987. Rudin uses the term "convex" to describe functions which are concave up.

We can show, using the definition, that $f(x) = x^2$ is concave up on any interval. Let $a, b \in \mathbb{R}$ and let $t \in (0, 1)$. Then

$$\begin{aligned}
f(ta + (1-t)b) &\geq tf(a) + (1-t)f(b) \\
(ta + (1-t)b)^2 &\geq ta^2 + (1-t)b^2 \\
t^2a^2 + 2t(1-t)ab + (1-t)^2b^2 &\geq ta^2 + (1-t)b^2 \\
(t^2 - t)a^2 + 2t(1-t)ab + ((1-t)^2 - (1-t))b^2 &\geq 0 \\
-t(1-t)a^2 + 2t(1-t)ab - t(1-t)b^2 &\geq 0
\end{aligned}$$

Since $0 < t < 1$, we can divide both sides by $-t$ and $1 - t$.

$$\begin{aligned}
a^2 - 2ab + b^2 &\geq 0 \\
(a-b)^2 &\geq 0.
\end{aligned}$$

The last inequality is always true for all a and b, and since each step is reversible, we have the established $f(ta+(1-t)b) \geq tf(a)+(1-t)f(b)$. This proves $f(x) = x^2$ is concave up everywhere. Once more, however, this is too much for the typical caluclus student. Better may be a translation into more common words. In English, that definition says the following.

> A continuous function $f(x)$ is concave up on an interval I if, and only if, for any $a, b \in I$, the values of the function on $[a, b]$ do not exceed the values of the line segment connecting $f(a)$ and $f(b)$. If the values of the line segment connecting $f(a)$ and $f(b)$ do not exceed the values of the function on $[a, b]$, the function is concave down.

This implies a picture: a secant line between two points on $f(x) = x^2$ will always be above the graph of the parabola. However, notice that the explicit use of the closed interval and the phrase "does not exceed" implies that points of inflection may be included in intervals of concavity.[7] Thus, if a function has a point of inflection at $x = 7$ and is determined to be concave up for $x > 7$, then we may correctly say that the function is concave up on $[7, \infty)$. Again, you can see why the AP Exam Readers are given leniency concerning brackets or parentheses!

Domains. The most vital consideration between brackets or parentheses is the issue of domains. Simply put, if the function is continuous at the endpoint of an interval of increasing/decreasing, or concavity, I urge my students to use a bracket. Otherwise, it gets a parenthesis. For example, consider the function $j(x) = x\ln(x)$. The derivative given by the product rule is $j'(x) = \ln(x) + 1$. Setting $j'(x) = 0$ yields $x = 1/e$. Thus, $j'(x) < 0$ for $0 < x < 1/e$ and $j'(x) > 0$ for $x > 1/e$. Since $j(x)$ is not defined for $x \leq 0$, the interval must be open on the left of our decreasing interval: $(0, 1/e]$. We also have $j(x)$ is increasing on $[1/e, \infty)$.

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[7] If $(a, f(a))$ is a point of inflection, then the segment from $f(a)$ to $f(b)$ is larger than f everywhere between a and b, except at $x = a$ where the values are equal. Hence, the values of the line do not exceed the function.

Conclusion. Should we use brackets or parentheses? The AP Exam generally accepts either—as long as the stated interval does not contradict the domain of the function—so students shouldn't worry about it on the AP Exam. But we should be teaching good mathematics, not just test preparation. The reliance on the first derivative test gives a students a false impression that "increasing" and "the derivative is positive" are identical properties when they are not. This leads to the confusion concerning brackets or parentheses. Moreover, this could also lead to misconceptions in later calculus courses, or could lead to incorrect answers with stranger functions (like the piecewise example above). Second, you must decide for yourself and your students whether to insist on brackets or parentheses. I personally insist my students use brackets whenever possible (but I accept either on assessments). This forces them to think about the domain of the function and reduces the likelihood that they will confuse an open interval with a point. Also, in mathematics, we generally strive to use the most inclusive definition possible. Including the endpoints when possible is the more inclusive approach.

2020 AP Calculus AB Exam

Marshall Ransom, Georgia Southern University

In 2020 before spring had even sprung, schools across the country were transitioning to online instruction in response to the COVID virus. The College Board and ETS decided that administration of the standard multiple choice section and six free response questions for the AP Calculus Exam was not going to be feasible, face-to-face. In addition, the reading could not be held safely face-to-face. Plans were made and many Test Development Committee (TDC) members and others at the College Board crafted and proofed and re-proofed a two question exam and put into place the methods and procedures by which the exam would be both administered and scored online.

This article will discuss procedures and issues related to the scoring of the 2020 AP Calculus Exam. Since no questions and scoring guidelines have been officially released, this article will refer to "mock" questions. These questions and "mock" scoring guidelines were prepared by Tom Dick of Oregon State, former TDC member and long time reader and leader and Stephen Kokoska of Bloomsburg University, former chief reader, continuing reader and leader. Two of these questions were referenced during a TI in Focus webinar broadcast, "Calculus Resources: Results, Technology and Content" September 15, 2020. More information regarding this webinar may be found at https://education.ti.com/en/professional-development/teachers-and-teams/online-learning/on-demand-webinars

Procedures:

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Question leaders and "Early Table Leaders" met online for six days before the actual scoring, finalizing the scoring guidelines and preparing responses with scoring commentaries that were used in training readers when they started the virtual Reading. After training, the uploaded student responses were available for readers to view and score as part of a robust software platform. This allowed readers to submit scores and ask questions of table leaders. It also allowed table leaders to backread previously scored exams and make comments to readers. Furthermore, table leaders, question leaders and exam leaders had a platform through which to deal with responses that were either difficult to score or bordering on illegible. On a personal note, I for one was impressed with the solid software online that made fair and consistent scoring possible.

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One question was introduced with information including a graph of a continuous function or the derivative of a function. The other question was introduced with information contained in a table of values. Additional given information involved such things as a function defined as an integral of a previously introduced function. A few topics were not considered on this exam in deference to the difficulties and time restrictions teachers and students faced in completing the course appropriately in the online environment imposed quickly upon schools nationwide. These were a few of the later topics outlined in the Course and Exam Description. For example, on the AB test there were no questions regarding volumes, although this did appear on the BC test.

The quick change to a two question exam gave rise to a scoring system different than the traditional 0 to 9 points for a free response question. For example, the one question (and its variations) that I worked on during most of the reading was worth 16 points. This was a question #1 for AB students. Those students also had a question #2 worth 12 points. The BC test was scored 18 points for one question and 11 points for the second question.

The first question referenced below involved functions, their derivatives and questions about a function maximum and/or minimum, points of inflection, the Mean Value Theorem and/or the Intermediate Value Theorem, the fundamental theorem, and other basic AP Calculus concepts concerning function behaviors, with volume perhaps coming into play on the BC exam. The second question relying on values in a table was given in a context such as motion with time in seconds and distance in meters. This question allowed the context of the problem to enter into student solutions and explanations.

The "mock" questions and scoring guidelines appear at the end of this article. They will be referenced in the discussion of the scoring below. Note that these are worth more points than were on the actual exam. This is in an attempt to address a wide variety of topics, perhaps a bit more than students saw, but of the same type. A variety of questions related to the given information and graphs and tables for both the AB and BC exams show a strong attempt at offering a thorough and comprehensive AP Calculus Exam.

Question #1:

THE AB "MOCK" QUESTION #1:

(a) Critical points occur where the derivative is 0 or undefined. Since $f = g'$ is given as continuous, all that needs to be shown to earn this one point are the values of x where $f = 0$.

(b) To earn these points, a response needs to refer appropriately to the change in signs of g'. If the reference is to the function f then there needs to be the explicit link $f = g'$. If this link was not established, students were still eligible for 2 of the 3 points. Some students referred to signs over intervals. These intervals should be correct. For example, $f < 0$ for $0 < x < 2$ and $f > 0$ for $2 < x < 5$ could be used to justify a relative min at $x = 2$. But stating that $f < 0$ for $x < 2$ and $f > 0$ for $x > 2$ is not correct. This last statement would be interpreted as a global and not local argument for this *relative* minimum.

(c) Both correct intervals need to be stated. The scoring guideline provides a very straightforward way to provide reasoning. Many students connect concavity with the sign of the second derivative. In employing this reasoning, as much information as possible should be given in the student response. For example, a reference to

f' is only acceptable if the link $f = g'$ has been established. Any reference to signs of f' or g'' on intervals would require correct intervals and certainly not references to a sign at a point.

(d) & (e) Both values related to areas below the *x*-axis. In order to show the computation, an integral has to be shown. Part e is worth two points because the area involved is that of a rectangle less a quarter circle.

(f) To find absolute extrema on a closed interval, examine the values of the function at the endpoints and critical points. Arguments referring to changes in sign are very difficult to use correctly and sufficiently. Note that this question asks for the absolute maximum value of the function. Thus the value of x at that location does not answer the question. Calculate both endpoint values and any critical point value(s) and then state which function value is greatest.

(g) g'' is found from the slopes of f. This is only worth one point. Showing a correct calculation of the slope at the point where $x = 6$ is sufficient in order to earn this point.

(h) Students should be aware that the average rate of change of a function over an interval is part of the conclusion of the Mean Value Theorem. To invoke this theorem (which does not have to be named, but if so the name must be correct) both continuity and differentiability need to be referenced. If intervals are specified, they must be correct in accordance with the hypotheses of the MVT. The verbal description in the scoring guideline below is certainly correct. Also correct would be something like "such a d exists because $g'(d) = \frac{g(2) - g(0)}{2 - 0}$ where $0 < d < 2$."

(i) Since the numerator and denominator in the expression $\frac{3x + g(x)}{\sin(x)}$ are both continuous, it is a simple matter to state that the limits of both are 0. Recognizing the "form" $\frac{0}{0}$ earns the first point. What can lose this point is explicitly using an equal sign as in $\lim_{x \to 0} \frac{3x + g(x)}{\sin(x)} = \frac{0}{0}$. Calculating derivatives of both numerator and denominator earns the second point. Evaluating the limit of the resulting expression earns the answer point.

(j) Computation of h' requires use of both a product rule and the chain rule. Each is worth 1 point in scoring the computation. For example, "$h'(x) = 1 \bullet g\left(x^2\right) + g'\left(x^2\right) \bullet 2x$ would earn a point for the chain rule, but no point for a correct product rule. Evaluating this expression correctly at $x = \sqrt{2}$ earns the third point for this part.

Question #2:

THE AB "MOCK" QUESTION #2:

(a) A local linear approximation requires use of a tangent line near $x = 11.8$. This would be at $x = 12$ where the slope of the line is given by $y'(12) = -5$. Using an equation of this line earns the first point. Using $x = 11.8$ to correctly calculate the approximation earns the second point.

(b) Correctly evaluating the expression $\frac{y'(6) - y'(2)}{6 - 2}$ earns this point.

(c) $y''(4)$ requires an explanation in the context of meters for distance and seconds for time *at* the time $t = 4$. This does not relate to the calculation in part b, and does not occur over an interval, but at a specific time. Since this is the second derivative of the position function, it is acceleration. Units must state meters per second per second to describe acceleration in this context.

(d) The average value of acceleration over [0, 12] is given by $\frac{1}{12-0}\int_0^{12} a(t)dt = \frac{1}{12-0}\int_0^{12} y''(t)dt$. Showing this earns the first point. The antiderivative earns the second point. The answer point is earned by a correct evaluation.

(e) Calculating this Riemann sum requires use of three midpoint values of y', multiplying those by the width of the appropriate intervals, and adding them. This involves the sum of three products. For the last several years on the AP Calculus Exam, the point for this is awarded if all or all but one of the interval and function values are correct. The second point is only awarded for the correct answer, simplified or not.

(f) Since y' is velocity and this definite integral does not involve absolute value, the numerical result is the displacement of the particle over the given time interval. Units in this question are awarded a second point.

(g) The search is for three times where $y' = 0$. This is an intermediate value between times given in the table where the sign of y' changes. Conditions for applying the IVT exist because of the differentiability of y' which implies its continuity. This earns the first point. Further specification of times (in the scoring guideline referred to as *a*, *b* and *c*) and their locations earns the second point.

(h) Note that $y'' = (y')'$ so that the MVT can be applied to y'. Using the three zeros of y' established in part *g*, the MVT can be applied twice to guarantee that y'' is 0 between (taken 2 at a time) the zeros found in part *g* for y'. An alternate solution can be used because of the conditions already mentioned and that y' is 0 at the endpoints of two intervals. Thus Rolle's Theorem can be applied to guarantee that the derivative of y' is 0 somewhere on each of the two intervals.

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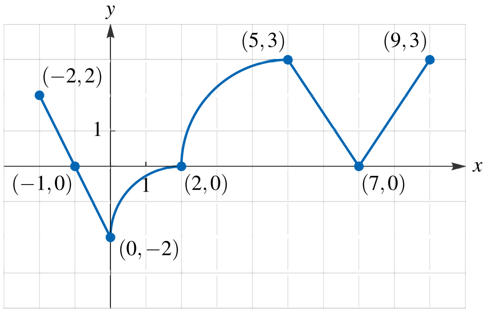
AP Calculus Mock Exam

AB 1

The continuous function f has domain $-2 \le x \le 9$. The graph of f, consisting of three line segments and two quarter circles, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_0^x f(t)\,dt$ for $-2 \le x \le 9$.

(a) Find the x-coordinate of each critical point of g on the interval $-2 \le x \le 9$.

(b) Classify each critical point from part (a) as the location of a relative minimum, a relative maximum, or neither for g. Justify your answers.

(c) For $-2 \le x \le 9$, on what open intervals is g increasing and concave down? Give a reason for your answer.

(d) Find the value of $g(-1)$. Show the computations that lead to your answer.

(e) Find the value of $g(2)$. Show the computations that lead to your answer.

(f) Find the absolute maximum value of g over the interval $-2 \le x \le 5$.

(g) Find the value of $g''(6)$, or explain why it does not exist.

(h) Must there exist a value of d, for $0 < d < 2$, such that $g'(d)$ is equal to the average rate of change of g over the interval $0 \le x \le 2$? Justify your answer.

(i) Find $\lim_{x \to 0} \frac{3x + g(x)}{\sin x}$. Show the computations that lead to your answer.

(j) The function h is defined by $h(x) = x \cdot g(x^2)$. Find $h'(\sqrt{2})$. Show the computations that lead to your answer.

<table>
<tr><th>Solution</th><th>Scoring</th></tr>
<tr><td>(a) $g'(x) = f(x) = 0 \quad \Rightarrow \quad x = -1,\ 2,\ 7$</td><td>1: answer</td></tr>
<tr><td>(b) At $x = -1$, g has a relative maximum because $g'(x) = f(x)$ changes from positive to negative there.
At $x = 2$, g has a relative minimum because $g'(x) = f(x)$ changes from negative to positive there.
At $x = 7$, g has neither because $g'(x) = f(x)$ does not change sign there.</td><td>3: 1 : $x = -1$ relative maximum with justification
1 : $x = 2$ relative minimum with justification
1 : $x = 7$ neither with justification</td></tr>
<tr><td>(c) g is increasing where $g' = f$ is positive.
g is concave down where $g' = f$ is decreasing.
g is increasing and concave down on the intervals $(-2, -1)$ and $(5, 7)$.</td><td>2 : 1 : answer
1 : reason</td></tr>
<tr><td>(d) $g(-1) = \int_0^{-1} f(t)\,dt$
$= -\int_{-1}^{0} f(t)\,dt = -\left(-\frac{1}{2}(1)(2)\right) = 1$</td><td>1 : answer</td></tr>
<tr><td>(e) $g(2) = \int_0^{2} f(t)\,dt$
$= -\left(2 \cdot 2 - \frac{1}{4} \cdot \pi \cdot 2^2\right) = -(4 - \pi)$</td><td>2 : 1 : area of quarter circle
1 : answer</td></tr>
<tr><td>(f) The absolute maximum value occurs at an endpoint of the interval or a critical point.
Consider a table of values.
$\begin{array}{r|l} x & g(x) \\ \hline -2 & 0 \\ -1 & 1 \\ 2 & \pi - 4 \\ 5 & \pi - 4 + \frac{1}{4}\pi 3^2 = \frac{13}{4}\pi - 4 \end{array}$
The absolute maximum value of g is $\frac{13}{4}\pi - 4$.</td><td>4: 1 : considers $x = -2$ and $x = 5$
1 : considers $x = -1$ and $x = 2$
1 : answer
1 : justification</td></tr>
</table>

Solution	Scoring
(g) $g''(6) = \frac{3-0}{5-7} = -\frac{3}{2}$	1 : answer
(h) $g' = f \;\; \Rightarrow \;\; g$ is differentiable on $0 < x < 2 \;\; \Rightarrow \;\; g$ is continuous on $0 \le x \le 2$ Therefore, the Mean Value Theorem can be applied to g on the interval $0 \le x \le 2$ to guarantee that there exists a value of d, for $0 < d < 2$, such that $g'(d)$ equals the average rate of change of g over the interval $0 \le x \le 2$.	$2 : \begin{cases} 1 : \text{conditions} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$
(i) $\lim_{x \to 0} (3x + g(x)) = 0$ $\lim_{x \to 0} \sin x = 0$ Therefore the limit $\lim_{x \to 0} \frac{3x + g(x)}{\sin x}$ is in the indeterminate form $\frac{0}{0}$ and L'Hospital's Rule can be applied. $\lim_{x \to 0} \frac{3x + g(x)}{\sin x} = \lim_{x \to 0} \frac{3 + g'(x)}{\cos x} = \frac{3 + g'(0)}{\cos 0}$ $= \frac{3 + f(0)}{\cos 0} = \frac{3 + -2}{1} = 1$	$3 : \begin{cases} 1 : \text{conditions for L'Hospital's Rule} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(j) $h'(x) = 1 \cdot g(x^2) + x \cdot g'(x^2) \cdot 2x$ $= g(x^2) + 2x^2 f(x^2)$ $h'(\sqrt{2}) = g(2) + 2 \cdot 2 \cdot f(2)$ $= (\pi - 4) + 4 \cdot 0 = \pi - 4$	$3 : \begin{cases} 1 : \text{product rule} \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2020 AP Calculus BC Exam

Marshall Ransom, Georgia Southern University

In 2020 before spring had even sprung, schools across the country were transitioning to online instruction in response to the COVID virus. The College Board and ETS decided that administration of the standard multiple choice section and six free response questions for the AP Calculus Exam was not going to be feasible, face-to-face. In addition, the reading could not be held safely face-to-face. Plans were made and many Test Development Committee (TDC) members and others at the College Board crafted and proofed and re-proofed a two question exam and put into place the methods and procedures by which the exam would be both administered and scored online.

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The quick change to a two question exam gave rise to a scoring system different than the traditional 0 to 9 points for a free response question. For example, the one question (and its variations) that I worked on during most of the reading was worth 16 points. This was a question #1 for AB students. Those students also had a question #2 worth 12 points. The BC test was scored 18 points for one question and 11 points for the second question.

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Question #1:

THE BC "MOCK" QUESTION #1:

(a) Critical points occur where $g'=0$ or does not exist. Because of the graph and more precisely because g is twice differentiable which implies that g' is continuous, we have that g' does exist. It can be seen that $g'=0$ where $x=9$, and this statement earns the first point. The second point is earned by declaring a relative max for g at this point because g' changes from positive to negative there.

(b) The function g has points of inflection where g' changes from increasing to decreasing or vice versa. Such a change takes place only where $x=4.2$. (Another way to see this is that the slopes of g', which give g'', change sign at this point.)

(c) This area is given by the definite integral of g' from 0 to 9. An antiderivative using the Fundamental Theorem of Calculus is *g*. The integral $\int_0^9 g'(x)dx$ must be shown to earn the first point. The antiderivative earns the second point and the correct evaluation of $g(9)-g(0)$ earns the third, answer point.

(d) The perimeter of this region includes the arc length of the curve from 0 to 9 and the lengths of the segments from 0 to 1 on the *y*-axis and 0 to 9 on the *x*-axis. The first point is awarded for a correct integral that calculates the arc length given by $\int_0^9 \sqrt{1+\left(\frac{d}{dx}(g')\right)^2}\,dx = \int_0^9 \sqrt{1+(g'')^2}\,dx$. The answer point is awarded for adding this to the lengths of the two segments.

(e) This asks for a value $g(c)=0$ between $g(0)$ and $g(9)$. If the function *g* is continuous, the IVT can be applied. For the first point, a student should state that *g* is continuous because *g* is differentiable. The second point is awarded if an interval shows 0 between the values of $g(0)=-7$ and $g(9)=12$.

(f) The first point is for using properties of integrals: the integral of a sum is the sum of the integrals and the contant passes freely to both terms. The second point is for a correct antiderivative of $\sqrt{x}$. The third point is for a correct calculation of the final answer.

(g) Since the numerator and denominator in the expression $\frac{x\cos(x)}{g(x)+2x+7}$ are both continuous, it is a simple matter to state that the limits of both are 0. Recognizing the "form" $\frac{0}{0}$ earns the first point. What can lose this point is explicitly stating that $\lim_{x\to 0}\frac{x\cos(x)}{g(x)+2x+7}=\frac{0}{0}$ with an equal sign. Calculating derivatives of both numerator and denominator earns the second point. Evaluating the limit of the resulting expression earns the answer point.

(h) The work shown in the scoring guidelines earns three points. A perhaps unusual teaching technique which I sometimes use is the following: $h'(x)=\frac{d}{dx}\int_{x^2}^{0} g(x)dx=\frac{d}{dx}\left(\text{Anti}\left(g(0)\right)\right)-\frac{d}{dx}\left(\text{Anti}\left(g\left(x^2\right)\right)\right)=0-g\left(x^2\right)2x$.

This would earn the first two points because the FTC has been applied and the $2x$ indicates application of the chain rule. A correct computation of this numerical value (using the given $g(9)=12$) earns the third point.

(i) The area $A(x)$ is that of a triangular cross section, $\frac{1}{2}(\text{base}\times\text{height})$. The height is given as *x* and the base of each triangle is the distance from the *x*-axis to the graph, which is g'. Thus $A(x)=\frac{1}{2}xg'(x)$ for one point.

(j) The volume is found by integrating the cross section areas from 0 to 9. This requires integration by parts. For the first point, the reader would be looking for $xg(x)-\int g(x)dx$. A correct evaluation of the expression from the IBP work earns the second point.

(k) The integrand $\frac{g''(x)}{g'(x)}$ is not defined at one point in the interval given because $g'(9)=0$. Thus one point is awarded for recognizing this as an improper integral and for using the appropriate limit notation. The second point is for the antiderivative $\ln|g'(x)|$. (Note: students should be trained to always show absolute value in the case of $\int \frac{du}{u} = \ln|u| + C$.) The evaluation needs to show limit notation and the result $-\infty$ or that the limit does not exist.

(l) The first point is earned for the second order Taylor polynomial, showing correct form of the coefficients and corresponding powers of *x*, using 2! appropriately. Using correct values of the derivatives of *g* in this expression, and adding the terms, earns the second point. If "$\cdots$" is added, the second point is not earned. Errors in simplification come off the second point. Only polynomials centered at 0 are eligible for any points.

Question #2:

THE BC "MOCK" QUESTION #2:

(a) Speed is the magnitude of the velocity. At $t = 6$ the velocity vector is $\left\langle [x'(6)]^2,\ [y'(6)]^2 \right\rangle$. This vector does not need to be shown, but its magnitude $\sqrt{[x'(6)]^2 + [y'(6)]^2}$ should be shown in order to earn the first point. Substituting correct values into this expression for both $x'(6)$ and $y'(6)$ earns the second point.

(b) Using the integral $\int_4^b x'(t)dt$ where $b = 12$ earns the first point. The choice of 12 for *b* is needed because the only *x*-coordinate position in the given information is at time $t = 12$: $x(12) = 4$. In order to find $x(4)$, one can consider the fact that $\int_4^{12} x'(t)dt = x(12) - x(4)$ or that the position $x(12) = x(4) + \int_4^{12} x'(t)dt$ (which is the initial position plus the net change in position over the interval $4 \le t \le 12$). Using either equation, it can be seen that $x(4) = x(12) - \int_4^{12} x'(t)dt$. To earn the answer point, the value of the integral must be computed and appropriately combined with $x(12) = 4$.

(c) The first point is earned for presenting the correct first step in Euler's method with at most one error. Tabular work can be shown, but an unlabeled table must be accompanied by the correct answer. If there is at most one error, the student earns the first point but is not eligible for the second point.

(d) One point is for calculation of the slope of this tangent line. Since information about this curve is provided separately for each of the coordinates of a point on the curve, the form $\frac{y'(12)}{x'(12)}$ should be used in student work. A correct form of a tangent line equation is needed in order to earn the second point.

(e) This question asks for the rate of change in distance, $r(t)$, between a particle on the curve and the Origin at the time $t = 12$. The distance between (0, 0) and $\left(x(t), y(t)\right)$ is given by $r(t) = \sqrt{x(t)^2 + y(t)^2}$. This earns the first point. The rate of change in distance is given by $r'(t) = \frac{2x(t)x'(t) + 2y(t)y'(t)}{2\sqrt{x(t)^2 + y(t)^2}}$ and earns the second point. Evaluating this last expression earns the answer point.

(f) This requires an appeal to the sign of the result for $r'(12)$. The correct answer is that the particle is moving further away from the Origin at time $t = 12$. An incorrect answer could not be justified in order to earn this point.

(g) A correct term in this polynomial is $\frac{y^{(p)}(12)(t-12)^p}{p!}$. Presenting two of these terms earns the first point. Adding all four correct terms including the values of $y^{(p)}(12)$ earns the second point. If terms are not centered at 12, no points are earned.

(h) Total distance traveled for $12 \le t \le 15$ is given by $\int_{12}^{15} (speed)\, dt = \int_{12}^{15} \sqrt{x'(t) + y'(t)}\, dt$. Given in the problem is an expression for $x'(t)$. It is stated in part h that $y(t)$ over this interval is given by the Taylor polynomial which was computed in part g. Thus $y'(t)$ is the derivative of the result from part g. A correct computation of $y'(t)$ earns the first point. Including this in the definite integral for speed over $12 \le t \le 15$ earns the second point.

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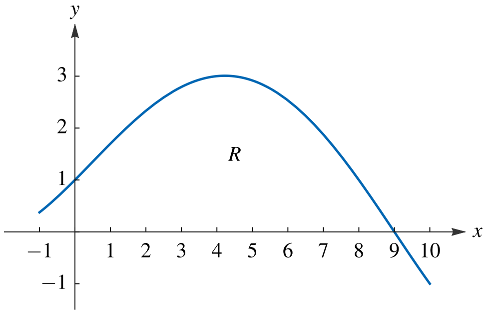
AP Calculus Mock Exam

BC 1

The graph of g', the derivative of the twice-differentiable function g, is shown for $-1 < x < 10$. The graph of g' has exactly one horizontal tangent line, at $x = 4.2$.



Graph of g'

Let R be the region in the first quadrant bounded by the graph of g' and the x-axis from $x = 0$ to $x = 9$. It is known that $g(0) = -7$, $g(9) = 12$, and $\int_0^9 g(x)\,dx = 27.6$.

(a) Find all values of x in the interval $-1 < x < 10$, if any, at which g has a critical point. Classify each critical point as the location of a relative minimum, relative maximum, or neither, Justify your answers.

(b) How many points of inflection does the graph of g have on the interval $-1 < x < 10$? Give a reason for your answer.

(c) Find the area of the region R.

(d) Write an expression that represents the perimeter of the region R. Do not evaluate this expression.

(e) Must there exist a value of c, for $0 < c < 9$, such that $g(c) = 0$? Justify your answer.

(f) Evaluate $\int_0^9 \left[\frac{1}{2}g(x) - \sqrt{x}\right] dx$. Show the computations that lead to your answer.

(g) Evaluate $\lim_{x\to 0} \dfrac{x \cos x}{g(x) + 2x + 7}$. Show the computations that lead to your answer.

(h) Let h be the function defined by $h(x) = \int_{x^2}^{0} g(t)\,dt$. Find $h'(3)$. Show the computations that lead to your answer.

(i) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis is a right triangle with height x and base in the region R. The volume of the solid is given by $\int_0^9 A(x)\,dx$. Write an expression for $A(x)$.

(j) Find the volume of the solid described in part (h). Show the computations that lead to your answer.

(k) Find the value of $\int_0^9 \dfrac{g''(x)}{g'(x)}\,dx$ or show that it does not exist.

(l) If $g''(0) = 0.7$, find the second degree Taylor polynomial for g about $x = 0$.

Solution	Scoring
(a) $g'(x) = 0$: $x = 9$ $g'(x)$ DNE: none g has a critical point at $x = 9$. At $x = 9$, g has a relative maximum because $g'(x)$ changes from positive to negative there.	2: $\begin{cases} 1 : \text{critical point at } x = 9 \\ 1 : \text{relative maximum with justification} \end{cases}$
(b) The graph of g has a point of inflection where g' changes from increasing to decreasing or from decreasing to increasing. g' changes from increasing to decreasing at $x = 4.2$. Therefore the graph of g has one point of inflection at the point where $x = 4.2$.	2: $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(c) Area $= \int_0^9 g'(x)\,dx = \Big[g(x)\Big]_0^9$ $= g(9) - g(0) = 12 - (-7) = 19$	3: $\begin{cases} 1 : \text{definite integral for area} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$
(d) $P = 1 + 9 + \int_0^9 \sqrt{1 + g''(x)^2}\,dx$	2: $\begin{cases} 1 : \text{definite integral} \\ 1 : \text{answer} \end{cases}$
(e) Since g is differentiable, then g is continuous on $0 \le x \le 9$. $g(0) = -7 < 0 < 12 = g(9)$ By the Intermediate Value Theorem, there exists a value of c, for $0 < c < 9$, such that $g(c) = 0$.	2: $\begin{cases} 1 : \text{conditions} \\ 1 : \text{conclusion using the Intermediate Value Theorem} \end{cases}$
(f) $\int_0^9 \left[\frac{1}{2}g(x) - \sqrt{x}\right] dx = \frac{1}{2}\int_0^9 g(x)\,dx - \int_0^9 \sqrt{x}\,dx$ $= \frac{1}{2}(27.6) - \left[\frac{2}{3}x^{3/2}\right]_0^9$ $= 13.8 - \frac{2}{3}(27)$ $= 13.8 - 18 = -4.2$	3: $\begin{cases} 1 : \text{properties of definite integrals} \\ 1 : \text{antiderivative of } \sqrt{x} \\ 1 : \text{answer} \end{cases}$

Solution	Scoring
(g) $\lim_{x\to 0}(x\cos x)=0$ $\lim_{x\to 0}(g(x)+2x+7)=0$ Therefore the limit $\lim_{x\to 0}\frac{x\cos x}{g(x)+2x+7}$ is in the indeterminate form $\frac{0}{0}$ and L'Hospital's Rule can be applied. $\lim_{x\to 0}\frac{x\cos x}{g(x)+2x+7}=\lim_{x\to 0}\frac{x\cdot(-\sin x)+1\cdot\cos x}{g'(x)+2}$ $=\frac{0\cdot(-\sin 0)+1\cdot\cos 0}{g'(0)+2}=\frac{1}{3}$	$3:\begin{cases}1:\text{conditions for L'Hospital's Rule}\\1:\text{applies L'Hospital's Rule}\\1:\text{answer}\end{cases}$
(h) $h'(x)=\frac{d}{dx}\left[\int_{x^2}^{0}g(t)\,dx\right]$ $=-\frac{d}{dx}\left[\int_{0}^{x^2}g(t)\,dt\right]$ $=-g(x^2)\cdot(2x)=-2xg(x^2)$ $h'(3)=-2\cdot 3\cdot g(9)=-6\cdot 12=-72$	$3:\begin{cases}1:\text{Fundamental Theorem of Calculus}\\1:\text{Chain Rule}\\1:\text{answer}\end{cases}$
(i) $A(x)$ represents the area of a right triangle at each x. $A(x)=\frac{1}{2}xg'(x)$	1 : answer
(j) $V=\int_0^9 A(x)\,dx=\frac{1}{2}\int_0^9 xg'(x)\,dx$ Use integration by parts. $u=x \qquad dv=g'(x)\,dx$ $du=dx \qquad v=\int g'(x)\,dx=g(x)$ $V=\frac{1}{2}\left(\Big[x\cdot g(x)\Big]_0^9-\int_0^9 g(x)\,dx\right)$ $=\frac{1}{2}\left([9\cdot g(9)-0\cdot g(0)]-27.6\right)$ $=\frac{1}{2}(9\cdot 12-27.6)=40.2$	$2:\begin{cases}1:\text{integration by parts}\\1:\text{answer}\end{cases}$

Solution	Scoring
(k) $\int_0^9 \frac{g''(x)}{g'(x)}\,dx = \lim_{t\to 9^-}\int_0^t \frac{g''(x)}{g'(x)}\,dx$ Let $u = g'(x)$, then $du = g''(x)\,dx$ and $dx = \frac{du}{g''(x)}$ $\int \frac{g''(x)}{g'(x)}\,dx = \int \frac{g''(x)}{u}\cdot\frac{du}{g''(x)} = \int \frac{du}{u}$ $= \ln\lvert u\rvert = \ln\lvert g'(x)\rvert$ $\lim_{t\to 9^-}\int_0^9 \frac{g''(x)}{g'(x)}\,dx = \lim_{t\to 9^-}\Big[\ln g'(x)\Big]_0^t$ $= \lim_{t\to 9^-}[\ln g'(t) - \ln g'(0)]$ $= \lim_{t\to 9^-}\ln g'(t) = -\infty$ Therefore the improper integral does not exist.	$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$
(l) $g(0) = -7, \quad g'(0) = 1, \quad g''(0) = 0.7$ $T_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2$ $= -7 + 1\cdot x + \frac{0.7}{2}\,x^2$ $= -7 + x + 0.35\,x^2$	$2 : \begin{cases} 1 : \text{form of } T_2(x) \\ 1 : \text{answer} \end{cases}$

AP Calculus Mock Exam

BC 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

A particle moves in the coordinate plane with position $(x(t), y(t))$ at time t, where t is measured in seconds and $x(t)$ and $y(t)$ are twice-differentiable functions, both measured in meters.

For all times t, the x-coordinate of the particle's position has derivative $x'(t) = \frac{t}{\sqrt{t^2 + 25}}$. Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \leq t \leq 12$ seconds are shown in the table.

The position of the particle at time $t = 12$ is $(x(12), y(12)) = (4, -3)$.

(a) Using correct units, find the speed of the particle at time $t = 6$.

(b) Find the exact value of $x(4)$, the x-coordinate of the position of the particle at time $t = 4$.

(c) Using Euler's method, starting at time $t = 12$ with 4 steps of equal size, approximate $y(4)$, the y-coordinate of the position of the particle at $t = 4$.

(d) Find an equation of the line tangent to the path of the particle at time $t = 12$.

(e) Let $r(t)$ be the distance between the particle and the origin $(0, 0)$ at time t. Find $r'(12)$.

(f) Is the particle moving closer or further from the origin at time $t = 12$? Justify your answer.

(g) Given $y''(12) = -2$ and $y'''(12) = 8$, find the third-degree Taylor polynomial approximation for y about $t = 12$.

(h) Suppose that over the time interval $[12, 15]$ the y-coordinate of the position of the particle is the same as the Taylor polynomial approximation found in part (g). Set up but do not evaluate an expression that represents the total distance traveled by the particle over the interval $[12, 15]$.

Solution	Scoring
(a) $x'(6) = \frac{6}{\sqrt{6^2 + 25}} = \frac{6}{\sqrt{61}}$ and $y'(6) = -2$ speed $= \sqrt{[x'(6)]^2 + [y'(6)]^2}$ $= \sqrt{\left(\frac{6}{\sqrt{61}}\right)^2 + (-2)^2}$ $= \sqrt{\frac{36}{61} + 4} = 2\sqrt{\frac{70}{61}}$ m/s	$2: \begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer with units} \end{cases}$
(b) $x(4) = x(12) - \int_4^{12} x'(t)\,dt$ $= (4) - \int_4^{12} \frac{t}{\sqrt{t^2 + 25}}\,dt$ $= (4) - \left[\sqrt{t^2 + 25}\right]_4^{12}$ $= (4) - [\sqrt{169} - \sqrt{41}] = \sqrt{41} - 9$	$3: \begin{cases} 1 : \text{integral} \\ 1 : \text{uses } x(12) \\ 1 : \text{answer} \end{cases}$
(c) $y(10) \approx y(12) + (-2) \cdot y'(12) = -3 + (-2)(-5) = 7$ $y(8) \approx y(10) + (-2) \cdot y'(10) = 7 + (-2)(-1) = 9$ $y(6) \approx y(8) + (-2) \cdot y'(8) = 9 + (-2)(3) = 3$ $y(4) \approx y(6) + (-2)y'(6) = 3 + (-2)(-2) = 7$	$2: \begin{cases} 1 : \text{First step in Euler's method} \\ 1 : \text{answer} \end{cases}$
(d) $\left.\frac{dy}{dx}\right\vert_{t=12} = \frac{y'(12)}{x'(12)} = \frac{-5}{\frac{12}{\sqrt{12^2 + 25}}} = -\frac{65}{12}$ An equation of the tangent line at the position $(x(12), y(12)) = (4, -3)$ is $y = -\frac{65}{12}(x - 4) - 3$	$2: \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

Solution	Scoring
(e) $r(t) = \sqrt{x(t)^2 + y(t)^2}$ $r'(t) = \frac{1}{2}(x(t)^2 + y(t)^2)^{-1/2}(2x(t)x'(t) + 2y(t)y'(t))$ $= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}$ $r'(12) = \frac{x(12)x'(12) + y(12)y'(12)}{\sqrt{x(12)^2 + y(12)^2}}$ $= \frac{(4)\left(\frac{12}{13}\right) + (-3)(-5)}{\sqrt{4^2 + (-3)^2}}$ $= \frac{243}{65}$	3: $\begin{cases} 1 : \text{expression for } r(t) \\ 1 : \text{expression for } r'(t) \\ 1 : \text{answer} \end{cases}$
(f) $r'(12) = \frac{243}{65} > 0$ The distance r to the origin is increasing at time $t = 12$. Therefore the particle is moving away from the origin at time $t = 12$ seconds.	1 : answer with reason
(g) Let $g(t)$ be the third degree Taylor polynomial for y at $t = 12$. $g(t)$ $= y(12) + y'(12)(t-12) + \frac{y''(12)}{2!}(t-12)^2 + \frac{y'''(12)}{3!}(t-12)^3$ $= -3 + (-5)(t-12) + \frac{-2}{2}(t-12)^2 + \frac{8}{6}(t-12)^3$ $= 3 - 5(t-12) - (t-12)^2 + \frac{4}{3}(t-12)^3$	2: $\begin{cases} 1 : \text{two terms of the Taylor polynomial} \\ 1 : \text{remaining terms} \end{cases}$
(h) For $12 \le t \le 15$, $y(t) = g(t) = 3 - 5(t-12) - (t-12)^2 + \frac{4}{3}(t-12)^3$ $x'(t) = \frac{t}{\sqrt{t^2+25}}$ $y'(t) = g'(t) = -5 - 2(t-12) + 4(t-12)^2$ The distance traveled by the particle over the time interval $[12, 15]$ is $\int_{12}^{15} \sqrt{\left[\frac{t}{\sqrt{t^2+25}}\right]^2 + \left[-5 - 2(t-12) + 4(t-12)^2\right]^2}\, dt$	2: $\begin{cases} 1 : \text{expression for } y'(t) \\ 1 : \text{expression for distance traveled} \end{cases}$