CSI: Calculus/Statistics Insider

Official Newsletter of the Georgia Association of Advanced Placement Math Teachers

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GA²PMT Constitution Revisions

Revision has been posted on the wiki site called "GAAPMT Group Information." Please take a moment to review the changes to the constitution.

Technology Workshops

Workshops focusing on TI-Nspire™ include choice of the TI-Nspire CX or TI-Nspire CX CAS

Pricing:

2-Day, 3-Day, and Virtual Workshops: $350* with technology; $300* without technology

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Atlanta, GA July 14 - July 16, 2014 (Two Choices)

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Platform: TI-Nspire™ CX handhelds and TI-Nspire™ Navigator™

Subject: High School Mathematics

This workshop is designed for educators who need support in transitioning to the Common Core State Standards for high school Algebra, Geometry, and Statistics. It covers basic features of the TI-Nspire™ CX handheld and TI-Nspire™ Teacher Software, with an emphasis on building pedagogical skills for leading discussions and engaging students in the Mathematical Practices.

The TI-Nspire CX handheld is introduced in the context of dynamic, interactive lessons aligned to the CCSS.

- **Day One identifies main content support needs with coverage of key CCSS domains and clusters in Algebra, Geometry, and Statistics. Educators will be introduced to the Common Core Math Practices, along with essential technology skills for the TI-Nspire™ handheld and Teacher Software.**
- **Day Two includes broader coverage of technology-enhanced lessons with opportunities for differentiation based on teacher needs.**
- **Day Three includes deeper discussions about the role of technology in the Common Core Math Practices.**

At the conclusion of the workshop, participants should be ready to use TI-Nspire technology to deepen students' understanding of key CCSS topics in Algebra, Geometry and Statistics.

Lesson and Assessment Design with TI-NspireTM Technology

Platform: TI-Nspire™ CX handhelds and TI-Nspire™ Teacher Software

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Registration Link:
https://ww2.eventrebels.com/er/Custom/Firm16553/Firm16553LandingPage.jsp

2014 CollegeBoard Summer Institutes in Georgia

Date	AP Calculus AB	AP Calculus BC	AP Statistics
June 2	Woodward Academy, College Park	Woodward Academy, College Park	Woodward Academy, College Park
June 16	Kennesaw State University, Kennesaw Marist School, Atlanta		
June 23	Walton High School, Marietta	Marist School, Atlanta Walton High School, Marietta	Walton High School, Marietta
July 7	Marist School, Atlanta University of GA, Athens	University of GA, Athens Kennesaw State University, Kennesaw	University of GA, Athens
July 14			Woodward Academy, College Park

Check out additional resources at our

GA^{2}PMT Wiki – http://gaapmt.wikispaces.com

- ✓ Our bi-annual newsletters are on the Newsletter link.
- ✓ GA^{2}PMT presentations
- ✓ News about the annual conference and other events is on the Professional Development page.
- ✓ Links to other internet-based resources can be found in Electronic Resources
- ✓ Find connections to other professional organizations
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GA^{2}PMT Membership Form

(Oct. 1, 2013 – Sept. 30, 2014)

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2013 AP Calculus Free Response Commentary

2013 AB1
Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given the function $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is in hours and $0 \le t \le 8$ on a certain workday. $G(t)$ models the rate at which unprocessed gravel arrives at a gravel processing plant. At the beginning of this 8 hour workday, the plant has 500 tons of unprocessed gravel. Gravel is processed at a constant rate of 100 tons per hour. Part a asked for $G'(5)$ and an interpretation of this, using correct units, in the context of the problem. Part b asked for the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday. Part c asks whether the amount of unprocessed gravel at the plant is increasing or decreasing at $t = 5$ hours and requires work that leads to that answer. Finally, in part d students were asked to find and justify the finding of the maximum amount of unprocessed gravel at the plant during the hours of operation.

Part a:

There were two points for this part. The first point is computational (and this is a calculator active question). $-25\sin\left(\frac{25}{18}\right)$, -24.587 or -24.588 or equivalents were acceptable. Work was not necessary to show. Although some students calculated by hand; incorrect starting and stopping of that work was ignored if the student arrived at a correct result. The second point required interpretation, units and time. Therefore something equivalent to "at $t = 5$" and not any form of "during" a time interval had to be present in the students' answers. Students had to correctly explain that this number is the rate at which the rate of arrival is changing in tons per hour per hour.

Part b:

Two points were possible in this part. The first point is conceptual, showing that $\int_0^8 G(t)\,dt$ is needed to answer this question. The second point is for the answer. Some students showed an indefinite integral which only earned the second point in the presence of a correct answer. Some students showed the correct answer of 825.551 but added 500 to it, thus earning only the first point, provided there was a definite integral.

Part c:

To earn both points here, students had to be comparing $G(t)$ to 100 and not to $100t$ or using $G'(t)$. In other words, this comparison or a correct numerical value for $G(5)-100$ had to be shown. Evidence of a comparison at $t=5$ could be shown by $G(5)-100=-1.859$ or by noting that $G(t)<100$ when $t>4.923$. It was also possible for a well-labeled graph to show this comparison. Students then had to conclude "decreasing" and no further explanation was required.

Part d:

The first point is for consideration of a point internal to the interval where the derivative of the amount changes sign. Since the derivative of the amount of gravel is $G(t)-100$, setting this equal to zero is evidence of the consideration. Calculating the integral $\int_0^{4.923}\left(G(x)-100\right)dx$ was essential for getting a correct answer and as reference for the third, justification, point $\int_0^{8}\left(G(x)-100\right)dx$ also had to be shown. The 500 could be added implicitly to the integral work to get the second, answer, point. Some students launched into the calculation without showing how a 4.923 was obtained, but were still eligible for the second point. To justify the maximum, students could state that 4.923 was a "unique critical point", but also had to show a calculation at $t=8$. The value of 500 at the endpoint $t=0$ was given in the stem of the problem. This is a classic justification for an absolute maximum on a closed interval.

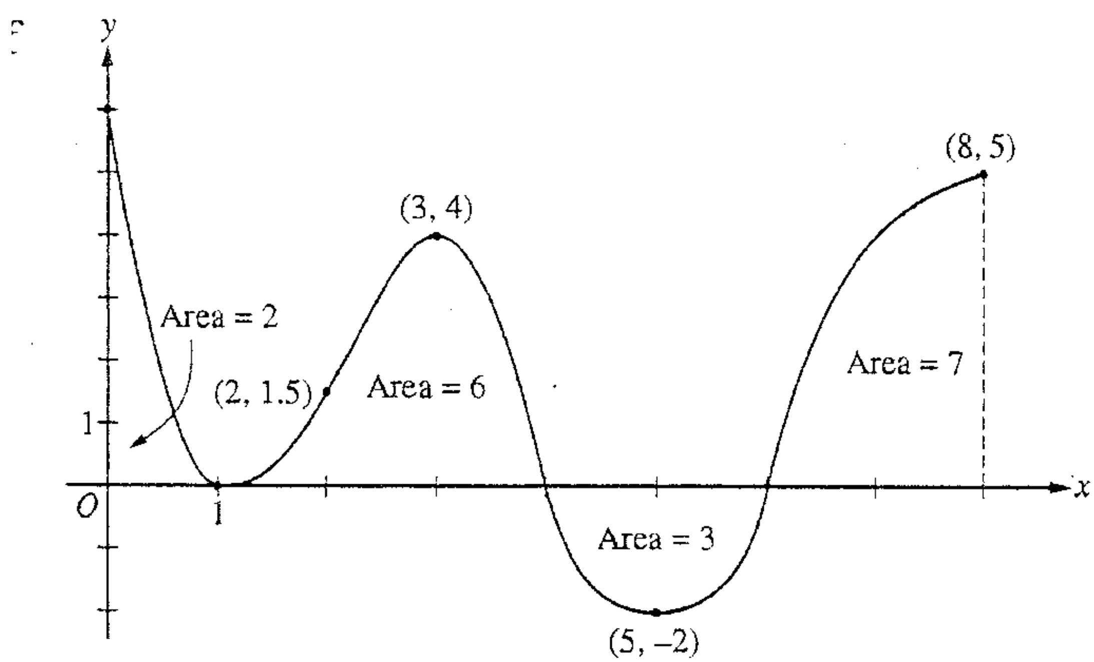
Observations and recommendations for teachers:

(1) Explaining the meaning (as in part a) of a rate of change clearly and completely is tough enough. When this is the rate of change of a rate of change, the explanation gets even more difficult. This needs to be practiced, along with correct and almost correct examples. One example that shows up sometimes in the "real world" has to do with inflation. Inflation is a positive rate of change in the consumer price index. What does it mean when the headline in a newspaper article reads, "Inflation Goes Down" ?
(2) A definite integral of a rate of change (part b) yields the net change in amount. Work with this as though it's a theorem, but one which should be dealt with in the context of real world situations.
(3) Arrival rate and departure rate (part c) are concepts for which examples from previous AP Calculus exams abound. Comparing these is the same as "setting the derivative equal to zero."
(4) The classic test for an absolute extremum on a closed interval is to calculate values at endpoints as well as at the location(s) of any relative extrema in the interval. Sometimes the calculations themselves are difficult, but the process remains the same and should be practiced both out of and in context of real world situations.
(5) Unusual units (part a) should be dealt with in the AP Calculus course. Students should be familiar with the idea of **per unit squared, although they should NOT substitute the word "acceleration" for "second derivative." I personally think that students who do this (and similarly use the word "velocity" out of context) have been given mostly distance, velocity and acceleration problems for practice with units and not other types of units in context. Students dealt with the units "tons per hour per hour" in many ways, many of which were incorrect. It could be noted that "$(\text{tons/hour})\times\text{hr}^{-1}$" is correct, although a student resorting to this presentation of units is indeed struggling.

2013 AB4
Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given the graph (below) of a function $f'(x)$ on the interval $0 \le x \le 8$ and told that $f(x)$ is twice differentiable. It is given that the graph of $f'(x)$ has horizontal tangents at $x = 1$, 3 & 5. The function $f(x)$ is defined for all real numbers, and $f(8) = 4$.



Graph of f'

In part a students are asked to find all values of x on $0 < x < 8$ for which the function f has a local minimum and to justify the answer. Part b is to determine the absolute minimum value of f on $0 \le x \le 8$ and to justify the answer. Part c asks for open intervals on $0 < x < 8$ for which the graph of f is both concave down and increasing, with an explanation of the reasoning. Part d defines function $g(x) = \left(f(x)\right)^3$, gives the information that $f(3) = -\frac{5}{2}$, and asks for the slope of the line tangent to the graph of g at $x = 3$.

Part a:

Students had to identify $x = 6$ as the only critical point where $f'(x)$ changes from negative to positive. This statement was worth one point. Vague justifications such as "the derivative changes (or the slope changes) from negative to positive" were not accepted. However (second derivative test), "a local min at $x = 6$ because $f'(6) = 0$ AND $f''(6) > 0$" was acceptable.

Part b:

The first of 3 points was for considering both $x = 0$ and $x = 6$, which could be shown by using $f(0)$ and $f(6)$ or ordered pairs containing 0 and 6, even in the presence of considering other values of x. The second point is still possible for students who don't earn the first point. The value $f(0) = -8$ is the answer. This can be found using $f(0) - f(8) = -\int_0^8 f'(x)\,dx$ (or some variation of that) and the areas given in the graph. Students were only eligible for the justification point if both of the first two points had been earned. Minimally, students had to be correctly comparing the values of $f(0)$ and $f(6)$ or (since $f(0)$ has already been stated to earn the second point) comparing $f(6)$ to $f(8)$.

Part c:

The correct intervals $0 < x < 1$ and $3 < x < 4$ (with or without endpoints) earned the first point. To earn the second point, the sign of $f'(x)$ had to be addressed as positive AND either $f'(x)$ decreasing or a statement that $f''(x) < 0$. A good explanation with only one of the intervals could earn the second point only, provided no additional incorrect intervals were included. Including incorrect intervals took students out of both points.

Part d:

The statement that $g'(3) = 3\left(f(3)\right)^2 f'(3) = 75$ earned all 3 points in part d. A few students used a product rule on $g(x) = f(x)f(x)f(x)$ or a quotient-product rule on $\dfrac{g(x)}{f(x)} = f(x)f(x)$. A chain rule error in calculating the derivative took the student out of both derivative points. An erroneous exponent or perhaps a 2 instead of a 3 in setting up the power rule only cost the students one point and allowed them to be eligible for an answer point, consistent with their work. Readers also saw students using the tangent line to the graph of f at $x = 3$ in place of the function $f(x)$. These students were working with $g(x) = \left[-\dfrac{5}{2} + 4(x-3)\right]^3$. The correct derivative for this function is $12\left(-\dfrac{29}{2} + 4x\right)^2$ which when evaluated at $x = 3$ is also 75. Without any explanation of why this works, students were not awarded any points because this is not the derivative of the function in the stem of the problem or more simply $-\dfrac{29}{2} + 4x \neq g(x)$. (This method works because the tangent line is the first two terms of the Taylor series for $g(x)$ centered at $x = 3$, and the other terms go to 0 at $x = 3$). I never saw any students explain why this method works.

Observations and recommendations for teachers:

(1) Regarding part a, students should know (and those taking the AP test often do) that local extrema occur at points where the derivative changes sign. Specifically, for a minimum, the sign change must be from negative to positive. Stating this using intervals where the derivative is either positive or negative can lead students to say something more global about the function and something that is not correct. Stating this in a vague manner such as saying "slope changes" is never a good idea. The specific function must be used in the statement. In classwork, locating such points should be accompanied by practice putting this into words, explaining what is happening that guarantees either a local maximum or minimum.

(2) Being able to write $f(b) - f(a) = \int_a^b f'(x)\,dx$ in various forms is an important way to gain information about a function from its derivative. For example, $f(b) = f(a) + \int_a^b f'(x)\,dx$ is also a useful form. In part b, we see the need sometimes to use a negative of the integral. Finding a definite integral in terms of areas in a graph of a derivative should be practiced, integrating both from left to right and right to left.

(3) The concepts of increasing/decreasing and concave up/down are valid on intervals, not at specific points. Identifying such behavior at specific points would take the students out of the justification point in part c. Justification should appeal to the derivative being positive/negative for increasing/decreasing on an interval. Similarly, concavity up/down can be justified by saying that the derivative (be specific about which derivative and which function!!!) is increasing/decreasing or that the second derivative is positive/negative.

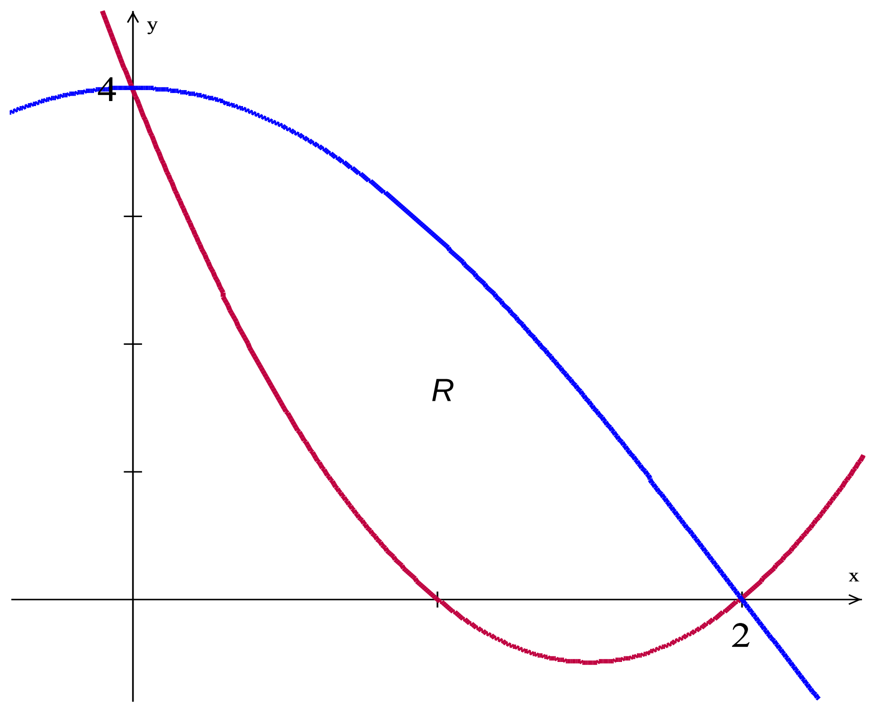
(4) When calculating a derivative on the AP test, chain rule errors are regularly treated as serious. Practice should be with both forms such as $\frac{d}{dx}\left(3x + \sin(4x)\right)^4$ as well as $\frac{d}{dx}\left(g(x)\right)^4$. In grading many papers, it appears to me that students are much more comfortable with the first form rather than the second. I humbly believe that this is a matter of more practice needed with a function name such as f or g rather than always with specific expressions in terms of x.

(5) Students need to be reminded not to simplify answers on the written portion of the exam. In part d, many students handled the $-\frac{5}{2}$ incorrectly, losing a point that had already been earned if they had avoided the temptation to "simplify."

2013 AB5
Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given the graphs (below) of $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. R is the region bounded by the two graphs.



In part a students are asked to find the area of R. Part b required students to set up, but not evaluate, an integral that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$. Using R as the base of a solid, part c asked for the volume of the solid formed if cross sections perpendicular to the x-axis are squares. Part c required setup only of an integral expression, not an evaluation.

Part a:

This part was worth 4 points, the first point for an integral setup for calculating area. In the presence of correct anti-derivatives, we still needed to see the integral in order to award the student the first point. We are looking for a classic "Top minus Bottom" but a reversal of this also earned the first point. Some students were intent on separating that portion of the graph below the x-axis. Thus correct variations of the form

$\int_0^1 \left(g(x) - f(x)\right) dx + \int_1^2 g(x)\,dx + \int_1^2 \left|f(x)\right| dx$ were also acceptable for the first point.

There was one point for each antiderivative, the sine and the cubic. Multiple errors in any one of these only resulted in the loss of that one point. A sign error in the sine cost the students the point, but a sign error in an otherwise correct cubic did not. Eligibility for the answer point required at least one of the two antiderivative points earned. If there was an error in the sine term, there had to be evidence of consideration of the chain rule; usually the presence of π was sufficient to meet this eligibility standard. In the case of a "correct" reversal, a student could calculate a negative area and subsequently make it positive. Other negative areas, perhaps resulting from a sign error in the cubic, could not earn the answer point.

Part b:

In part b there were two integrand points and one point for correct limits and constant. We needed to see radii squared. One was worth the first point. The second radius, correctly subtracted from the first, earned the students the second point. A reversal on the subtraction cost the students this second point. To be eligible for the third point, students had to earn at least one of the integrand points. Some students earned all three points, but continued working and made an algebra error which cost them one point.

Part c:

There was one point for a correct integrand, and order of subtraction did not matter because of the square. But correct use of parentheses did make a difference. If students did not earn the integrand point, they were not eligible for the limits and constant point (the constant here is 1). Importation of bad limits from part b (for example 0 to 4) already cost the students a point in part b; therefore, this was allowed for a student to earn the limits/constant point in part c.

Observations and recommendations for teachers:

(1) In finding the area bounded by two curves (part a), the only thing that matters is "Top minus Bottom" (or "Right minus Left") which is the length of an incremental rectangle. It matters not whether the region is ever above or below the x-axis. Thus length times width becomes "Top minus Bottom" times dx (or "Right minus Left" times dy). Students who tried to deal separately with the region below the x-axis had difficulties. Use of $g(x) - f(x)$ was appropriate since these functions were defined in

the stem of the problem. Students using the expressions in terms of x often made parentheses errors from which they did not recover.

(2) Students are expected to know basic u-substitution techniques, but part a did not require much. In fact, I want my students to know that $\int \cos(kx)\,dx = \frac{\sin(kx)}{k} + C$ without showing any u-substitution work. To generalize, $\int \text{trig}(kx)\,dx = \frac{\text{anti-trig}(kx)}{k} + C$ does not need any supporting u-substitution work after students have learned basic techniques of u-substitution work.

(3) Volumes of solids formed by cross sections have appeared on recent AP tests. The integrand is the area of the cross section, and dx or dy represents the "thickness" of the cross section. This should be practiced with square, semi-circle, triangle, and other rectangle possibilities for the cross section.

(4) Rotating a region about a line $y = k$ or $x = k$ away from a coordinate system axis should be practiced. The radii for this washer method are typically $f(x) - k$ or $g(y) - k$, and then they are squared.

(5) In more than one instance, improper use of parentheses hurt students on this problem. This should be addressed very specifically in both practice work and grading of that work. Suggestion: for the third or fourth set of parentheses, perhaps use brackets.

2013 AB 6
Al Wolmer, Yeshiva Atlanta High School

Problem overview: Students were presented with the differential equation

$$\frac{dy}{dx} = e^{y}\left(3x^2 - 6x\right)$$

and were told that $y = f(x)$ is the particular solution passing through the point $(1, 0)$. There were two parts to this problem: one where the student had to find the tangent line to the function at the given point and then use it to approximate the function value at a nearby point, and a second part, in which the student had to find the particular solution to the differential equation passing through the given point. The first part required the student to understand the concept of linearizations, the interpretation of a derivative as the slope of a tangent line, and how to evaluate the derivative at a point using both x and y coordinates. The second part required the student to solve the differential equation using separation of variables. The student had to find antiderivatives of both polynomial and exponential functions, include an arbitrary constant of integration, evaluate the constant using the initial condition, and finally, write the solution in its entirety.

Part (a):

This part was worth 3 points, the first point for evaluating $\frac{dy}{dx}$ at the point $(x,y)=(1,0)$.

The second point was for an equation of the tangent line and, finally, one point was for approximating $f(1.2)$ by using the tangent line.

The first point could be earned by explicitly writing the value of the derivative in simplified or unsimplified form : $\frac{dy}{dx}=e^0\left(3\cdot 1^2-6\cdot 1\right)$.

Simply writing the value of the slope of the line, i.e. $m=-3$, did not alone earn the point. However, it could be earned with the second point.

The second point was awarded for an equation of the correct tangent line, which could be presented in simplified, e.g., slope-intercept form: $y=-3x+3$ or unsimplified form, e.g., point-slope: $y-0=-3(x-1)$.

If the first point was not previously earned, but if the student made it clear in this part that the slope of the tangent line was found by evaluating the derivative at $(1,0)$ the first point was now earned as well. If the value of the slope was incorrect but the resulting incorrect tangent line went through $(1,0)$ this point was still earned and the student was eligible for the third point. However, if the correct point-slope form of the line was incorrectly simplified, the first two points were earned, but the third point would not be awarded.

The third point was for the correct approximation to the function. The student had to explicitly show that they were using the value $x=1.2$ by either substituting that value in the equation of the tangent line or by directly writing $y=-3(1.2-1)$ or $y=-3(1.2)+3$.

However, an answer such as $y=-3.6+3$ did not earn the point because it did not demonstrate the use of the tangent line equation.

Part (b):

This part was worth 6 points as follows: one for separation of variables, one for each of two antiderivatives, one for using an arbitrary constant of integration, one for using the initial condition that the particular solution goes through the point $(1,0)$ to evaluate the constant, and one for the correct form of the particular solution.

The first point was earned by correct separation. Various forms were accepted, e.g.,

$e^{-y}dy=\left(3x^2-6x\right)dx$ or $\int \frac{1}{e^y}\,dy=\int\left(3x^2-6x\right)dx$.

The first point (separation) was awarded even if (only) one of the differentials was missing or if the parentheses were missing. If both differentials were missing, the point could still be earned through correct antidifferentiation in the next two points. However, incorrect separations such as $e^y dy=\left(3x^2-6x\right)dx$ earned zero points here and for this entire part.

The second and third points were awarded for the correct antiderivatives, $-e^{-y}$ and x^3-3x^2.
If the antiderivatives were incorrect, there was an eligibility requirement for the fourth point.
Eligible forms were $\pm e^{-y} =$ "cubic – polynomial" for consideration for the fourth point.

The fourth point was awarded for appropriately including an arbitrary constant of integration.

The fifth point was awarded for using the initial condition to evaluate the constant of integration. The student had to explicitly show they were substituting $(x, y) = (1, 0)$. The correct value was $C = 1$. Incorrect values resulted in loss of the answer point (Point 6).

The sixth point was awarded for a correct form of the solution, e.g.,

$$y = -\ln\left(-x^3+3x^2-1\right),$$
$$y = \ln\left(\frac{1}{-x^3+3x^2-1}\right),$$
$$y = \ln\left(\frac{-1}{x^3-3x^2+1}\right), etc.$$

The sixth point was not awarded if parentheses were missing as in $y = -\ln -x^3+3x^2-1$
or if absolute value signs were included as in $y = -\ln\left|-x^3+3x^2-1\right|$.

Observations and recommendations for teachers:

1. Students should be trained regarding arithmetic computations to not do them unless absolutely necessary but, if necessary, to be explicit and show all work. In Part (a), numerically correct values of the slope were not awarded points if it was not clear that the derivative was being used. Of course, numerically incorrect values resulting from incorrect (and unnecessary) simplification were also not awarded points.
2. Even though leaving the differential off the end of an integral does not, in and of itself, incur a penalty, it is simply bad form and can lead to incorrect antiderivatives.
3. Far too many students incorrectly integrated as follows: $\int \frac{1}{e^y}\,dy = \ln\left|e^y\right| = y$.

Students were confusing $\int \frac{du}{f(u)}$ with $\int \frac{du}{u}$.

Teachers should take the time to reinforce the correct understanding of this.
4. Either when finding the antiderivative of the exponential or when solving for y in the very last point, many students incorrectly added absolute value signs, e.g.

$$y = -\ln\left|-x^3+3x^2-1\right|$$

losing one or more points, depending on where it occurred. This is not just an arbitrary penalty, because inclusion of the absolute value signs denotes a fundamental misunderstanding. The expression for y is found by extracting logarithms from the result

$$-e^{-y} = x^3-3x^2+1$$

and not from integration. This, along with the previous error, should be reviewed in class and the correct concept and procedures reinforced.

Introducing Series with Taylor Polynomials
by Dennis Wilson

If you will kindly pardon the pun, I have always been displeased by the *discontinuity* of my AP Calculus BC course. Upon finishing the unit on Differential equations, the course seemingly steps away from the concepts of calculus to explore infinite series and tests for their convergence. I always promise my students that we will be returning to the calculus they have grown to love after we learn these convergence tests.

Following conversations with Westminster's Nurfatimah Merchant and Georgia Southern's Marshall Ransom at Rock Eagle, I decided to escape this sequence dictated by my textbook, and introduce Taylor polynomials before discussing convergence tests. My students finish our unit on differential equations by approximating the base of the natural logarithm using Euler's Method. Moving from the "broken lines" of Euler's method to the curves of Taylor polynomials seems a logical next step to my students.

Starting cautiously with the linear approximation to $f(x) = e^x$ at $x = 0$, we next build a quadratic approximation and then proceed to polynomials of higher degrees. As these polynomials provide a higher degree of accuracy for e, my students become convinced of the validity of this method for our chosen function. The question then becomes, is the method valid for all functions.

We continue to explore other functions using the Taylor polynomial, namely $f(x) = \sin x$ and $f(x) = \cos x$. The Taylor polynomials for these functions continue to produce highly accurate approximations. Opportunities to connect these polynomials with differentiation and trigonometric identities are then explored. Students are now satisfied with the utility of Taylor polynomials and their ability to provide higher degrees of approximations for various tasks. The next function we explore though, creates significant challenges to their satisfaction.

Finding a Taylor Polynomial for the natural logarithmic function seems a logical next step, but it presents some difficulties previously not encountered. First, all the Taylor polynomials studied thus far were centered at $x = 0$. The fact that $f(x) = \ln x$ is undefined at $x = 0$ obviously makes this center impossible. After finding a Taylor polynomial for a known value of the natural logarithm at $x = 1$, the difficulty of the domain restrictions must again be considered. While $f(x) = \ln x$ is defined only for $x > 0$, our newly created polynomial is defined for all values of x.

$$\ln(x) \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

By exploring the approximations generated by higher degrees of polynomials at selected values of x, students are able to discover the idea of divergence and convergence for Taylor series.

Using diagrams like those shown in the given figures, students are able to visualize Taylor series by seeing each term and its effect on the partial sum. The lengths of the arrows in the diagrams represent the magnitude of the individual terms, while the tip of the arrow ends at the partial sum.

The first value outside the domain of $f(x) = \ln x$ that we explore is $x = -1$.

$$-2 - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} - \frac{2^5}{5} - \frac{2^6}{6} - \frac{2^7}{7} - \frac{2^8}{8} - \frac{2^9}{9} - \frac{2^{10}}{10} - \cdots$$

By calculating successive terms, students can see that each one is larger in magnitude than the previous. The ever increasing magnitude of the terms leads to an ever increasing magnitude of our Taylor polynomial's approximation, which is easily seen in the diagram 1. This series provides the basis for my introduction of the Divergence Test to students.

The next value we explore is $x = 0$. The series of term generated by our polynomial approximation is the harmonic series itself.

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \cdots$$

Although this series passes the Divergence Test, students are shown using various methods why it diverges. This series provides the basis for my introduction of the Integral Test and the p-series test.

The question now arises that if these values outside the domain of our original function diverge, what about values of x that are inside the domain. The first value we explore from inside the domain is $x = 1/2$. A careful term by term analysis of this series presents two opportunities.

$$-\left(\frac{1}{1}\cdot\frac{1}{2^1}\right) - \left(\frac{1}{2}\cdot\frac{1}{2^2}\right) - \left(\frac{1}{3}\cdot\frac{1}{2^3}\right) - \left(\frac{1}{4}\cdot\frac{1}{2^4}\right) - \left(\frac{1}{5}\cdot\frac{1}{2^5}\right) - \cdots$$

The first is a reintroduction of the geometric series. As you can see from the expanded form of the series, each term is just a fraction of the series the students first learn in Zeno's paradox. Students know that the geometric series has a limiting value, and therefore our series should have a limiting value as well since each term is less than its counterpart in the geometric series. This series provides the basis for my introduction of the comparison test.

The series for $x = 3/2$ provides two points of emphasis for continued discussion of the Taylor polynomial for $f(x) = \ln x$.

$$\left(\frac{1}{1}\cdot\frac{1}{2^1}\right) - \left(\frac{1}{2}\cdot\frac{1}{2^2}\right) + \left(\frac{1}{3}\cdot\frac{1}{2^3}\right) - \left(\frac{1}{4}\cdot\frac{1}{2^4}\right) + \left(\frac{1}{5}\cdot\frac{1}{2^5}\right) - \cdots$$

This series has the same terms as the series for $x = 1/2$, except that the signs alternate. The Comparison Test can be reapplied to show that a limiting value exists for this series as well. Also, the nature of the alternating series can be seen in the diagram 4 . Students are able to visualize how adding successive terms alternate the partial sums above and below the limiting value of the series. This is important as we consider the value of $x = 2$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

The alternating behavior of the partial sum in this series is even more apparent in diagram 5. The values can be seen converging on to some limiting value. This series provides the basis for my introduction to the Alternating Series Test.

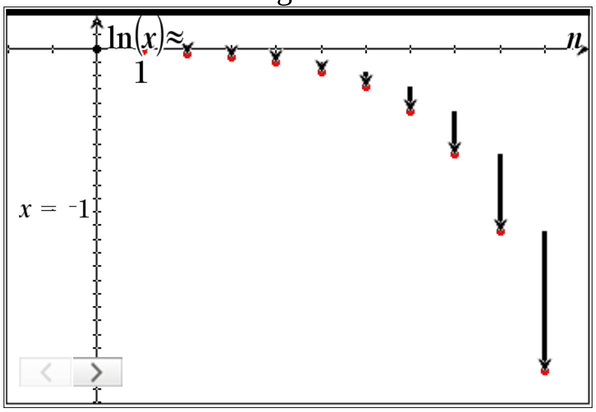
The last value we explore for with our Taylor polynomial probably provides students with their greatest consternation. The value of $x = 3$ is easily in the domain of $f(x) = \ln x$, but as student consider the terms, they see that the series of values fails to pass the Divergence Test.

$$2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5} - \frac{2^6}{6} + \frac{2^7}{7} - \frac{2^8}{8} + \frac{2^9}{9} - \frac{2^{10}}{10} - \cdots$$

Using diagram 6, students once again see that no limiting value exists for the series. The fact that the Taylor Polynomial fails to provide an approximation for values inside the domain of our function makes students question our previous results with $f(x) = e^x$, $f(x) = \sin x$, and $f(x) = \cos x$. The dilemma of knowing when a Taylor Polynomial will approximate a function must be considered. This motivates the introduction of the Ratio Test and the Radius of Convergence.

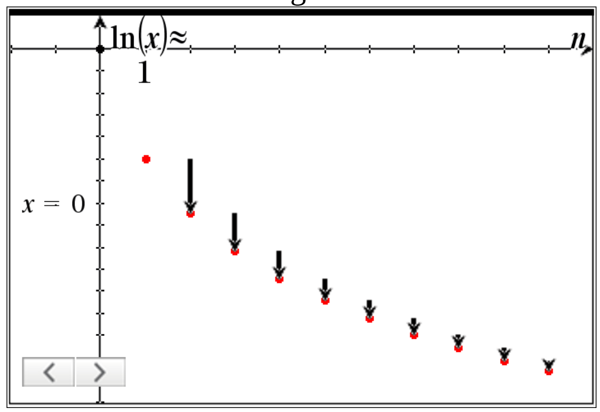
While this new sequence of topics has led to hopscotching among the sections of my text for appropriate exercises, the benefits are quite clear. The first benefit is the elimination of what students considered an apparently meaningless break in the presentation of material. The continuity of the course is now preserved. The primary benefit though is that it gives students a concrete example of the use of each convergence test and its importance. Students are no longer blindly learning convergence test, but they are now empowered with visual confirmation for each series convergence.

Diagram 1



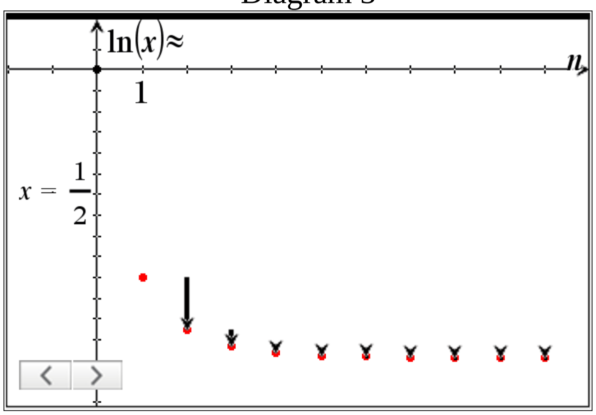
Series for $x = -1$

Diagram 2



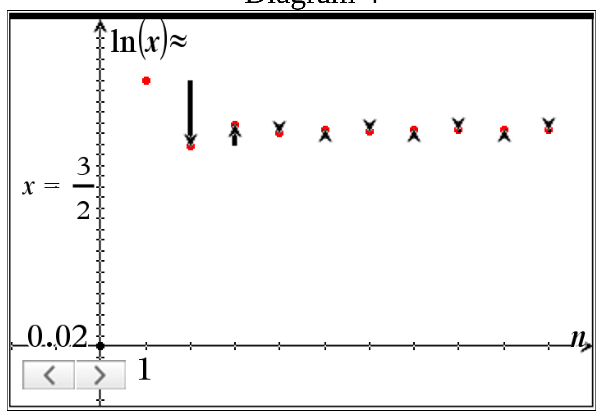
Series for $x = 0$

Diagram 3



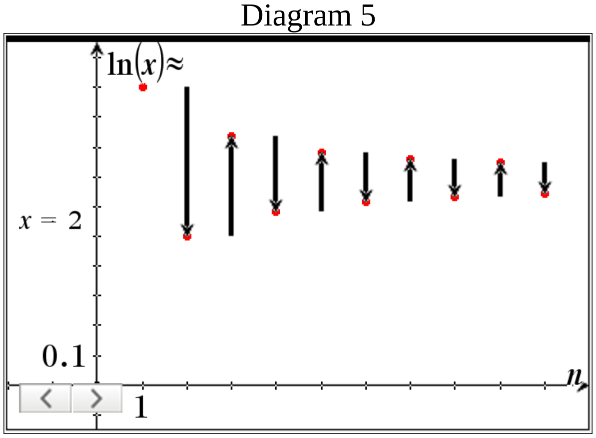
Series for $x = 1/2$

Diagram 4



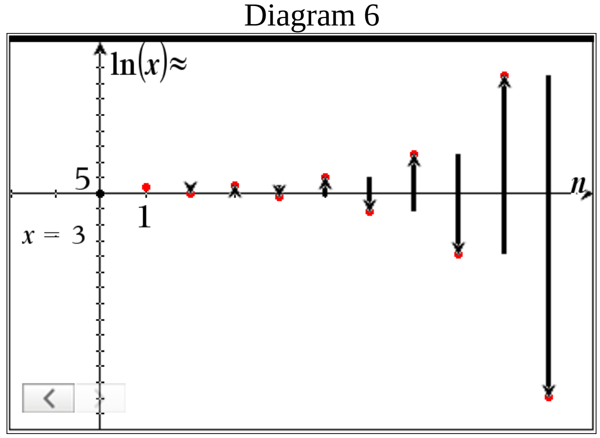
Series for $x = 3/2$

Diagram 5



Series for $x = 2$

Diagram 6



Series for $x = 3$

Solving Some (Possibly AP) Calculus Problems
GCTM Conference 2013
Presenters: Marshall Ransom & Chuck Garner

Objectives:

EXPECT TO:

1. Do more calculus than you will listen to

2. Reinforce an appreciation for the difficulties students encounter when learning calculus and solving problems

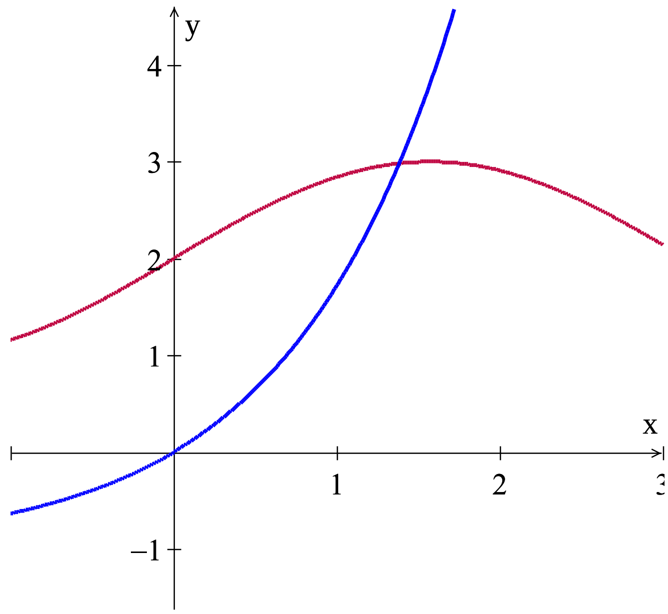
3. Discuss a variety of calculus topics in the context of solving problems

4. When possible, put solutions into the context of how the problems might be graded on the AP Calculus Exam

5. Provide ideas for the writing of problems and problem-solving in your teaching

(1) A region is bounded by the y-axis, $y = e^x - 1$, and $y = \sin(x) + 2$, as shown in the graph at right. (Units are cm).

(a) Calculate the area of this region.



(b) A solid is formed in the region bounded by the two curves $y = e^x - 1$ and $y = \sin(x) + 2$ and the line $y = 2$. This solid is formed by cross-sections that are squares perpendicular to the y-axis. Calculate the volume of this solid.

(c) A line $y = k$ where $k > 2$ passes horizontally through the region. Set up an equation involving k which if solved would calculate the specific value of k that would divide the volume found in part b in half.

(NOTE: Increasingly, we have seen on the AP test a differential equation in a real world context).

(2) The cooling system in my old truck holds about 10 liters of coolant. Last summer I flushed the system by running tap water into a tap-in on the heater hose while the engine was running and was simultaneously draining the thoroughly mixed fluid from the bottom of the radiator. Water flowed in at about the same rate as the mixture flowed out. This was about 2 liters per minute. The mixture was initially about 50% antifreeze. If we let W be the amount of water in the coolant after t minutes then it follows that

$$\frac{dW}{dt} = 2 - 2\left(\frac{W}{10}\right)$$

(a) Explain why the expression for $\frac{dW}{dt}$ is correct.

(b) Find W as a function of time.

(c) How long should I have let the water run into the system to ensure that the mixture was 95% water?

(3) Let $F(x)$ and $F'(x)$ be continuous and assume values in the table below. Use $F(x)$, $F'(x)$ and the values in the table to answer the questions below.

x	–1	0	1	2	3	4	5
$F(x)$	18	0	–12	0	54	168	360
$F'(x)$	–15	–11	0	30	81	150	237

(a) What is the slope of a line tangent to the graph of $F(x)$ at $x = 0$?

(b) Write an equation of a line tangent to $F(x)$ at $x = 4$.

(c) Why can we not determine whether $F(x)$ is increasing on the interval (2, 3)?

(d) Calculate the value of $\frac{d}{dx}\left(x^3 \bullet F(x)\right)$ at $x = 1$.

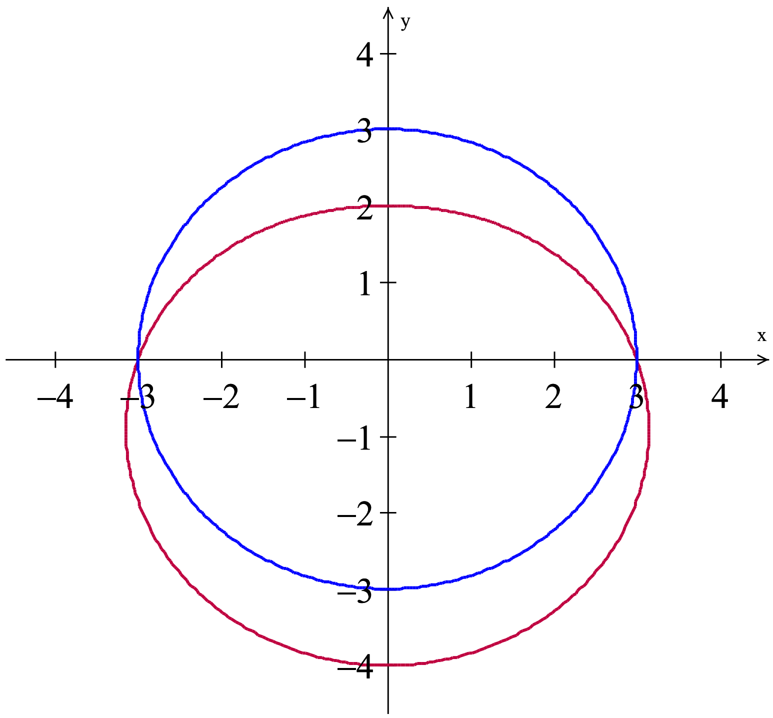
(e) Write and solve your own problem requiring the quotient rule.

(f) Calculate the value of $\int_3^5 F'(x)\,dx$.

(g) Write and solve your own integral problem requiring use of $\tan^{-1}(x)$.

(h) If $H(x) = \int_{x^2}^{2} F(t)\,dt$, calculate $H'(1)$.

(4) The BC question (NOT calculator active): Graphs of the polar curves $r = 3 - \sin(\theta)$ and $r = 3$ are shown in the figure at right. Answer the following questions.



(a) A particle moves along the curve $r = 3 - \sin(\theta)$ such that at any time t in seconds, $\theta = t$. What is the position vector of this particle in terms of t? What are the exact rectangular coordinates of the position when $t = \frac{\pi}{6}$?

(b) What is the exact value of the slope of a line tangent to the curve $r = 3 - \sin(\theta)$ where $\theta = \frac{\pi}{6}$?

(c) Write an equation of the line tangent to $r = 3 - \sin(\theta)$ at the point where $\theta = \frac{\pi}{6}$. Use this equation to estimate the y-coordinate of the point where $x = x_0 - 0.01$, x_0 being your x-coordinate from part a.

(d) What is the area of the region above the x-axis, below the circle and above the limaçon?

(e) The length of the curve $r = 3 - \sin(\theta)$ above the x-axis from points A(3, 0) to B(–3, 0) is approximately 7.7253. What is the approximate average speed of the particle as it traverses this arc starting at $t = 0$?

(4) The AB question (this is NOT calculator active):
Two particles are in motion on the x-axis. The position of particle P is given by $p(t) = 3\sin\left(\frac{\pi}{2}t\right)$ and the position of particle Q is given by $q(t) = t^3 - 2t^2 + t - 3$ for any time $t \geq 0$.

(a) How far apart are the particles at time $t = 0$? Write, but do not evaluate, an expression for the average distance between the particles on the interval $0 \leq t \leq 4$.

(b) On the interval $0 \leq t \leq 3$ find all times t when particle Q is moving to the left.

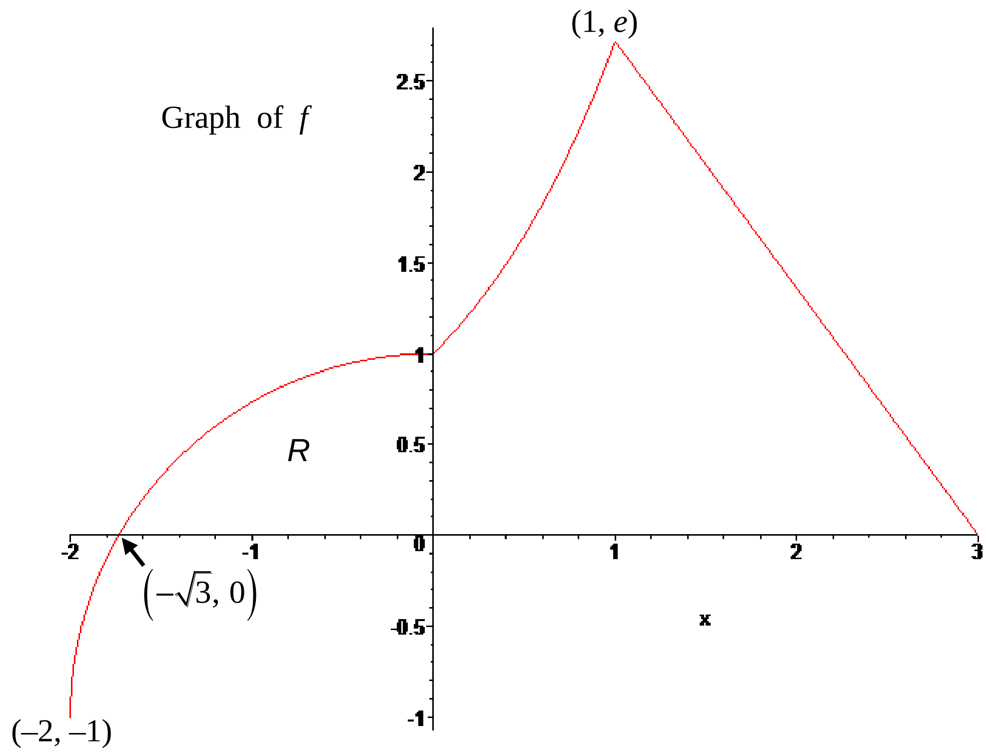
(c) On the interval $0 \leq t \leq 3$ find all times when the particles are moving in the same direction.

(d) What is the velocity of particle Q at time $t = 2$? Is the particle Q speeding up, slowing down, or neither of these at the time $t = 2$? Justify your answer.

(e) Find the total distance traveled by particle P on the interval $0 \leq t \leq 2$. Reminder: this is not a calculator active problem.

(5) The continuous function f is defined for $-2 \le x \le 3$. The graph of f consists of a quarter circle, e^x, and one line segment as shown at right. Answer the following questions using the fact that $g(x) = \int_0^x f(t)\,dt$.

(a) Calculate the values of $g(1)$ and $g(3)$.



(b) The region R is the portion of the quarter circle above the x-axis. The exact area of this region is given by $-\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$. Using this value, calculate the exact value of $g\left(-\sqrt{3}\right)$.

(c) Explain why $g\left(-\sqrt{3}\right)$ is an absolute minimum value for $g(x)$.

(d) Find the average rate of change of f on the interval $0 \le x \le 3$. There is no value $c \in (0,3)$ for which $f'(c)$ equals that average value. Explain why this does not contradict the Mean Value Theorem.

(e) What is the value of $g''(2)$?

(f) For what value(s) of x does the graph of g have a point of inflection? Justify your answer(s).

Solutions and Comments

(1) Note: This problem would be a calculator active problem, both to find the intersection point and to calculate the numerical value of the integral. For example, the calculation of an anti-derivative in part b would involve integration by parts (a BC topic) and some rather creative persistence. The point of intersection of these curves will sometimes be awarded one point on the AP Calculus Exam if used in work which proceeds to answering the questions. This point is $(x, y) = (1.38183\ldots, 2.98219\ldots)$. Remember, 3 digit accuracy is all that is required on the AP Calculus exam. In the solutions below, We will refer to these numbers as A = 1.38183… and B = 2.98219…, using A and B in the integrals.

(a) Area = $\int_0^A \left(\sin(x)+2-\left(e^x-1\right)\right)dx \approx 1.975$.

(b) (On the AP test, students sometimes must show some work in producing x in terms of y.) The values of x are: $x=\ln(y+1)$ and $x=\sin^{-1}(y-2)$. The first value of x shown is the curve on the right.

Volume = $\int_2^B (Right-Left)^2\,dy = \int_2^B \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy \approx 0.5475...$

(c) For a k which finds half the volume, use $\int_2^k \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy \approx \frac{1}{2}(0.5475)$

OR use $\int_k^B \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy \approx \frac{1}{2}(0.5475)$.

<u>Variations on parts b & c:</u> (b) Use semi-circles and (i) calculate the volume directly and (ii) calculate the volume using an appropriate fraction of your answer to the square cross-section problem.
(c) Assume for $k > 2$, the line $y = k$ is moving upward at a rate of 1.5 cm/min. When $k = 2.5$ at what rate is the volume using square cross-sections increasing (region bounded below by $y = 2$ and above by $y = k$)?
Express your answer using proper units.

Variation (b): Using semi-circular cross-sections:

(i) $r=\frac{1}{2}\left(\ln(y+1)-\sin^{-1}(y-2)\right) \to r^2=\left(\frac{1}{2}\left(\ln(y+1)-\sin^{-1}(y-2)\right)\right)^2$

Therefore the volume $=\int_2^B \frac{1}{2}\pi r^2\,dy = \frac{\pi}{2}\int_2^B \left(\frac{1}{2}\left(\ln(y+1)-\sin^{-1}(y-2)\right)\right)^2 dy \approx 0.215$

(ii) OR fractional method: The ratio of the area of a semi-circle with radius r to the area of a square of side $2r$ is $\frac{\pi}{8}$. The volume is $\frac{\pi}{8}\times 0.5475 \approx 0.215$.

Variation (c): We know that $V = \int_2^k \left(\ln(y+1) - \sin^{-1}(y-2)\right)^2 dy$. Therefore,

$$\frac{dV}{dt} = \left(\ln(k+1) - \sin^{-1}(k-2)\right)^2 \times \frac{dk}{dt} \rightarrow \frac{dV}{dt} = \left(\ln(2.5+1) - \sin^{-1}(2.5-2)\right)^2 \times 1.5 \quad \text{cm}^3/\text{min}$$

(2) (a) The rate of change in the amount of water is water inflow rate minus water outflow rate. The water inflow rate is 2 liters per minute. The fraction of water in the outflow is $\frac{W}{10}$ giving an outflow rate of $2 \times \frac{W}{10}$.

Therefore inflow rate minus outflow rate = $\frac{dW}{dt} = 2 - 2\left(\frac{W}{10}\right)$.

(b) We separate and integrate:

$$\frac{dW}{dt} = 2 - 2\left(\frac{W}{10}\right) = \frac{10-W}{5}$$

$$\int \frac{dW}{10-W} = \int \frac{1}{5}\,dt$$

$-\ln|10-W| = \frac{1}{5}t + C$ $\leftarrow$ on the AP test, this line alone usually gets at least 3 points

$10 - W = Ke^{-\frac{1}{5}t}$ $\leftarrow$ (What about absolute value?)

$$W = 10 - Ke^{-\frac{1}{5}t}$$

The initial condition is $W(0)$ = 50% water = 5 liters.

We get from the last line above $5 = 10 - Ke^{-\frac{1}{5}\bullet 0} \rightarrow K = 5$. Therefore $W(t) = 10 - 5e^{-\frac{1}{5}t}$.

(c)

$W(t) = 9.5 = 10 - 5e^{-\frac{1}{5}t} \rightarrow 0.5 = 5e^{-\frac{1}{5}t} \rightarrow \ln(0.1) = -\frac{1}{5}t \rightarrow t = -5\ln(0.1)$ minutes ≈ 11.513 minutes

(3) (a) $F'(0) = -11$

(b) $F(4) = 168$ and $F'(4) = 150 \;\rightarrow\; y - 168 = 150(x-4)$ (AP test: often, 1 point each for slope and equation)

(c) For $F(x)$ to be increasing on an interval, $F'(x)$ must be positive on the entire interval. In fact, we do not know from the table any values of $F'(x)$ on the open interval (2, 3).

(d) $\dfrac{d}{dx}\left(x^3 \bullet F(x)\right) = x^3 F'(x) + 3x^2 F(x)$. Therefore, at $x = 1$ we have $1^3 \bullet F'(1) + 3\left(1^2\right)F(1) = 0 + 3(-12)$
.

(e) Quotient Rule possible example: $\dfrac{d}{dx}\left(\dfrac{x^3}{F(x)}\right) = \dfrac{F(x) \bullet 3x^2 - x^3 F'(x)}{\left(F(x)\right)^2}$

Choose any value of x from the table and use appropriate $F(x)$ and $F'(x)$ values (beware of $F(x) = 0$).

(f) $\int_3^5 F'(x)\,dx = F(5) - F(3) = 360 - 54$

(g) Possible $\tan^{-1}(x)$ example: $\displaystyle\int_1^3 \frac{F'(x)\,dx}{1 + \left(F(x)\right)^2} = \tan^{-1}\left(F(3)\right) - \tan^{-1}\left(F(1)\right) = \tan^{-1}(54) - \tan^{-1}(-12)$

(h) $H(x) = \displaystyle\int_{x^2}^{2} F(t)\,dt \;\rightarrow\; H'(x) = -F\left(x^2\right) \bullet 2x \;\rightarrow\; H'(1) = -F\left(1^2\right) \bullet 2(1) = -(-12) \bullet 2$

(4) BC

(a) Because $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the position vector is $\left\langle 3\cos(t) - \cos(t)\sin(t),\, 3\sin(t) - \sin^2(t) \right\rangle$.

At $t = \frac{\pi}{6}$ we have

$x = 3\times\frac{\sqrt{3}}{2} - \frac{1}{2}\times\frac{\sqrt{3}}{2} =$ DO NOT SIMPLIFY (it's $\frac{5\sqrt{3}}{4}$) and $y = 3\times\frac{1}{2} - \left(\frac{1}{2}\right)^2$ OR $\frac{5}{4}$

(b) slope $= \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos(\theta) - 2\sin(\theta)\cos(\theta)}{-3\sin(\theta) - \left(\sin(\theta)(-\sin(\theta) + \cos(\theta)\cos(\theta)\right)}$

At $\theta = \frac{\pi}{6}$ we have a mess or (DO NOT SIMPLIFY) $-\frac{\sqrt{3}}{2}$

(c) $y - y_0 = m(x - x_0) \rightarrow y - \frac{5}{4} = -\frac{\sqrt{3}}{2}\left(x - \frac{5\sqrt{3}}{4}\right)$

At $x = \frac{5\sqrt{3}}{4} - 0.01$ we have $y - \frac{5}{4} = -\frac{\sqrt{3}}{2}\left(\left(\frac{5\sqrt{3}}{4} - 0.01\right) - \frac{5\sqrt{3}}{4}\right) \rightarrow y \approx \frac{\sqrt{3}}{2}(0.01) + \frac{5}{4}$

(d) Area = $\frac{1}{2}\int_0^{\pi} r_1^2\, d\theta - \frac{1}{2}\int_0^{\pi} r_2^2\, d\theta$ if $r_1 = 3$ and $r_2 = 3 - \sin(\theta)$.

$$\frac{1}{2}\int_0^{\pi}\left(3^2 - (3 - \sin(\theta))^2\right)d\theta = \frac{1}{2}\int_0^{\pi}\left(6\sin(\theta) - \sin^2(\theta)\right)d\theta = \frac{1}{2}\left(-6\cos(\theta) - \frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right)\right)\Bigg|_0^{\pi} = 6 - \frac{\pi}{4}$$

(e) The particle starts its motion at time $t = 0$ and reaches point B at time $t = \pi$. Thus the total time needed to traverse this arc is π seconds.

The average speed is $\frac{\text{total distance}}{\text{total time}} \approx \frac{7.7253}{\pi}$.

(4) AB

(a) The distance between the particles at time $t = 0$ is given by $|p(0) - q(0)| = |0 - (-3)| = 3$

The distance between the particles at any time t is given by $|p(t) - q(t)|$.

The average distance on $0 \le t \le 4$ is given by $\frac{1}{4}\int_0^4 |p(t) - q(t)|\,dt$.

(b) $q(t) = t^3 - 2t^2 + t - 3 \rightarrow q'(t) = 3t^2 - 4t + 1 = (3t-1)(t-1)$.

On $0 \le t \le 3$, particle Q is moving left when $v_q(t) = q'(t) < 0 \rightarrow \frac{1}{3} < t < 1$.

(c) On $0 \le t \le 3$, particle Q is moving right when $v_q(t) = q'(t) > 0 \rightarrow 0 < t < \frac{1}{3}$ or $1 < t < 3$.

$v_p(t) = p'(t) = \frac{3\pi}{2}\cos\left(\frac{\pi}{2}t\right) \rightarrow v_p(t) > 0$ for $0 < t < 1$ and $v_p(t) < 0$ for $1 < t < 3$.

Particles Q and P move in the same direction when $q'(t)$ and $p'(t)$ have the same sign. This occurs when $0 < t < \frac{1}{3}$.

(d) $v_q(2) = 3(2)^2 - 4(2) + 1 > 0$ and $a_q(2) = 6(2) - 4 > 0$. $\leftarrow$ this statement is sufficient justification

Since $v_q(2) \times a_q(2) > 0$, the motion of particle Q is speeding up at $t = 2$.

(e) $v_p(t) = p'(t) = \frac{3\pi}{2}\cos\left(\frac{\pi}{2}t\right)$. Total distance traveled on $0 < t < 2$ is given by $\int_0^2 |v_p(t)|\,dt$.

$$\int_0^2 |v_p(t)|\,dt = \int_0^1 \tfrac{3\pi}{2}\cos\left(\tfrac{\pi}{2}t\right)dt - \int_1^2 \tfrac{3\pi}{2}\cos\left(\tfrac{\pi}{2}t\right)dt = 3\sin\left(\tfrac{\pi}{2}t\right)\Big|_0^1 - 3\sin\left(\tfrac{\pi}{2}t\right)\Big|_1^2 = 6$$

(5)

(a) $g(1)=\int_0^1 e^t\,dt = e^1 - e^0 =$ DO NOT SIMPLIFY !!!

$$g(3)=\int_0^3 f(t)\,dt=\int_0^1 e^t\,dt+\int_1^3 f(t)\,dt=e^1-e^0+\frac{1}{2}\times 2\times e= \text{DO NOT SIMPLIFY !!!}$$

(b) We are given that $\int_{-\sqrt{3}}^{0} f(t)\,dt=-\frac{\sqrt{3}}{2}+\frac{2\pi}{3}$

$$g\left(-\sqrt{3}\right)=\int_0^{-\sqrt{3}} f(t)dt=-\int_{-\sqrt{3}}^{0} f(t)\,dt=-\left(-\frac{\sqrt{3}}{2}+\frac{2\pi}{3}\right)= \text{DO NOT SIMPLIFY !!!}$$

(c) $g'(x)=f(x)=0$ where $x=-\sqrt{3}$ (we do not have to consider $g'(x)=f(x)$ DNE because f is continuous, hence defined everywhere).

$g'(x)=f(x)$ changes sign from $-$ to $+$ at $x=-\sqrt{3}$ indicating a minimum value. Because $f(x)<0$ for $-2<x<-\sqrt{3}$ and $f(x)>0$ for $-\sqrt{3}<x<3$, this is an absolute minimum.

(d) Average rate of change is $\frac{f(3)-f(0)}{3-0}=-\frac{1}{3}$. For the MVT to apply, f must be differentiable at all points in the interval $(0, 3)$. However, f is not differentiable at $x = 1$.

(e) $g''(2)=f'(2)=$ slope of f at the point where $x=2 \rightarrow g''(2)=\frac{e-0}{1-3}=$ DO NOT SIMPLIFY !!!

(f) The graph of g has a point of inflection where $g''(x)=f'(x)=$ slope of f changes sign. This occurs at the point where $x = 1$.

(AP Exam grading and “correct” calculus note regarding part (f): all that is required for there to be a point of inflection is a change in sign of the second derivative. Unlike justifying a relative max or min, whether the change is from + to – vs. – to + is irrelevant).

THINGS I LEARNED TEACHING AP CALCULUS

CHUCK GARNER

I started teaching AP Calculus in 2002 at a magnet school for science and technology. I was not very good at it, which was a shame, since every student in my school had to take calculus to graduate, and I was the only calculus teacher. I realize now why I was not good at it. I taught calculus to my students the way I was taught calculus: heavy on symbolic manipulation and algebra skills; more concerned with finding antiderivatives than with understanding what the definite integral means; and only focused on those applications which could be approached in a rote fashion.

I have learned a lot since those first years teaching calculus. By focusing more on the conceptual rather than the computational, my students understand calculus better and have greater success on the AP exam. This approach helped my students, but it also helped me. I have stumbled across some interesting insights over the years, some developed over time and others gained in flashes of insight. You may already be familiar with much of what I have learned. But if you still teach calculus the way you were taught, you may learn something too. So indulge me as I present to you some of the things I have learned teaching AP Calculus.

The first thing I learned was that we should

KEEP THE ALGEBRA SIMPLE.

When I first started teaching AP Calculus, I went absolutely nuts trying to get kids to understand how to evaluate expressions like

$$\lim_{x\to 1}\frac{3x^3+7x^2-2x-8}{(x-1)^5} \quad \text{and} \quad \lim_{x\to 3}\frac{\sqrt{x+3}-\sqrt{6}}{x-3} \quad \text{and} \quad \int \frac{x^4-2}{x^4-2x^3+5x^2-8x+4}\,dx.$$

So much effort and frustration and time spent on the algebra! Even though most students recognized what to do, most students struggled to perform the task. But in studying the types of problems which actually appear on the AP Calculus Exam, I realized that no problems like these are tested. My guess as to why problems such as these do not appear is that students should not be inhibited from demonstrating knowledge of calculus because of the algebra.

But more than that, I would ask whether they are instructive at all. How can those three problems above be more instructive to students than

$$\lim_{x\to 2}\frac{x^2-4}{x-2} \qquad \text{or} \qquad \int \frac{4x^2}{x^2-4}\,dx,$$

two problems that do not require great amounts of tedious algebra. Besides, is this tedious algebra really necessary in our modern world? Particularly when many practicing scientists and engineers would use a computer to calculate anything which may involve such awful algebraic manipulations.

I can't believe how I made my students work at the beginning of each year. I covered the typical course sequence: those first two weeks of algebra review, followed by weeks of limits, before getting into the good stuff. And we require them to do crazy algebra to evaluate limits in order to illustrate *why* they absolutely had to have those algebra skills down! (As if we are saying

"See! Limits are an application of all those algebra skills! You must know algebra!") I spent so much time working with limits and getting kids to evaluate them using algebra tricks. However, once I examined the types of limit problems that are asked on the AP exam, I realized that

LIMITS ARE A WASTE OF TIME.

I know this is a controversial statement, but it is true: limits are a waste of time. Now, I'm not advocating the overthrow of established calculus hegemony. I realize the need for the epsilon-delta formulation as much as the next mathematician. However, the epsilon-delta form of limits has been long absent from the AP Calculus course, and the limit problems which remain all seem to fall into four categories: (1) direct evaluation; (2) left- and right-hand limits for piecewise functions; (3) definitions of continuity and the derivative; and all other limits can be solved with (4) l'Hôpital's rule.

The order of these four categories is the order in which I present these concepts about limits. I teach just enough about limits—through direct evaluation, piecewise functions, and functions whose graphs have holes—for students to understand the definition of the derivative. The key explanation which allows such an abbreviated treatment of limits is that I relate "limit" to the simple idea of "behavior." Thus, a limit describes a function's behavior near a point.

For example: What is the behavior of $f(x) = (x^2+2x)/x$ near $x = 0$? Since the function $g(x) = x+2$ is exactly equal to $f(x)$ for all $x \neq 2$, we may use g to model the behavior of f. Since the behavior of g near $x = 0$ is 2, the behavior of f near $x = 0$ must also be 2. That is, f is behaving like it will be 2 as x gets closer to zero. Thus, the limit of $f(x)$ as $x \to 0$ is 2.

After this, I continue with the standard series of lessons on differentiation, and conclude that with l'Hôpital's rule so students can handle all other kinds of limits, including limits involving infinity. Removing the need for all the ridiculous algebra has made my life and my students' lives much better, without sacrificing understanding. The massive amount of algebraic simplification has been reduced. This, along with the AP exam standard of not needing to simplify arithmetic, has lead me to a simple rule:

DON'T SIMPLIFY.

As AP Calculus teachers, we are already familiar with the fact that fractions need not be reduced. And even some arithmetic need not be performed. Certainly, however, some simplification is just never needed. I mean, why should anyone anywhere rationalize any denominator? (Can't math teachers agree that $\sin(\pi/4) = 1/\sqrt{2}$ is a perfectly good answer already?) Rationalization is nothing more than an antiquated holdover from the days before calculators. Let me explain: Notice that it is impossible to use long division to divide 1 by $\sqrt{2} = 1.41421356237\ldots$. But when you rationalize, then you can divide $\sqrt{2}$ by 2, to whatever degree of accuracy you need. But now we have calculators, and we no longer need to rationalize.

Even some algebraic simplification is unwarranted. Consider finding the domain of a function. If one simplifies first, then you may miss some values on the domain. This function appeared on the AP exam about 15 years ago:

$$f(x) = \sqrt{x^4 - 16x^2}$$

The first question asked the student to find the domain. If the student simplified it to $x\sqrt{x^2-16}$, then the domain is changed. (Not to mention the proper expression is $|x|\sqrt{x^2-16}$, but we can't expect all our students to be miracle workers!) The domain of the original function includes zero, but the domain of the simplified form excludes zero. This is a mistake, as the simplified form actually leads to a wrong answer! The correct domain is, of course, $(-\infty,-4]\cup\{0\}\cup[4,\infty)$.

Don't think simplifying expressions is never needed. Simplification is needed for those multiple-choice problems—students must match what they have obtained to the correct answer choice. In addition to algebra, students must be able to simplify expressions involving trigonometric functions, logarithmic functions, and exponential functions. (Which, by the way, is not strictly considered simplification. This would fall under *evaluating* trigonometric, logarithmic, and exponential functions, which must always be done, when it is possible to do so.) The logarithmic functions and their associated properties are particularly important. Unfortunately, many students, if they remember those "log properties" at all, do not know why they work or even why they exist. But there is a way to show them why, and that's by waiting to introduce them in calculus until integration. This way, you can

DEFINE THE LOGARITHM AS AN INTEGRAL.

What I present to my students—after teaching them the Fundamental Theorem—is a variation of the following. I define the function $L(x)$ as

$$L(x)=\int_1^x \frac{1}{t}\,dt.$$

I do not tell them that this is the logarithm; instead we (the class and I) will derive all the properties of the logarithm from this definition. This way, the properties have a reason for existing.

Notice that $L(x)$ cannot be defined for zero or negative values, because of the vertical asymptote of the function $f(t)=1/t$. This function L is defined in this way because its derivative needs to be $1/x$, which it is, according to the Fundamental Theorem:

$$L'(x)=\frac{d}{dx}\int_1^x \frac{1}{t}\,dt=\frac{1}{x}.$$

Hence, $L(x)$ is the antiderivative of $\frac{1}{x}$. Now we investigate properties of L. First, the class sees that

$$L(1)=\int_1^1 \frac{1}{t}\,dt=0.$$

Clearly, if $x>1$ then $L(x)$ is positive since we are calculating area under a curve. Also, if $0<x<1$, then, since the integral accumulates area in the opposite direction, $L(x)$ must be negative.

Is L the only function for which $1/x$ is its derivative? We answer this by finding the derivative of $L(kx)$ for constant k. By the chain rule,

$$L'(kx)=\frac{d}{dx}\int_1^{kx} \frac{1}{t}\,dt=\frac{1}{kx}\cdot k=\frac{1}{x}$$

so that $L(kx)$ is also an antiderivative of $1/x$. Hence, since two antiderivatives can at most differ by a constant, $L(kx)=L(x)+C$. We may find the value of C by letting $x=1$, where this becomes $L(k)=L(1)+C$. But $L(1)=0$, which implies $L(k)=C$. Therefore,

$$L(kx)=L(x)+L(k).$$

I believe this is the development of the "log property" $\log(ab)=\log a+\log b$ that makes the most sense to students at this level.

Then the class differentiates $L(x^p)$ for real p:

$$L'(x^p)=\frac{1}{x^p}\cdot px^{p-1}=p\cdot\frac{1}{x}=pL'(x).$$

So then $p(1/x)$ is the antiderivative of the two functions $L(x^p)$ and $p \cdot L(x)$. Hence, these functions must differ by a constant; this gives $L(x^p) = pL(x) + C$. Letting $x = 1$ results in $C = 0$. Therefore, in general,

$$L(x^p) = pL(x).$$

At this point, I finally define the function $L(x)$ as the logarithm function for my students, and point out that there must be some value of x which results in $L(x) = 1$. I simply say that whatever value that is we will call e.

Then we investigate functions of the form $f(x) = cL(x)$ for some nonzero constant c. All properties so far discovered must be satisfied; in particular, we want $cL(x) = 1$ for a particular value of x. Call this particular value b. Then b cannot be equal to 1 (since $L(1) = 0$), and so we have $cL(b) = 1$. Rewriting, we get $c = 1/L(b)$, so that $f(x) = L(x)/L(b)$. This implies that all functions which are multiples of $L(x)$ are simply $L(x)$ scaled by a particular value of L. Note that when this value is e, we have

$$\frac{L(x)}{L(b)} = \frac{L(x)}{L(e)} = \frac{L(x)}{1} = L(x),$$

which is the reason we term this "non-scaled" function the *natural logarithm*. Writing the natural logarithm of x as $\log_e x$ (to denote that the constant scale factor is $L(e)$), motivates us to define the other scaled functions of $L(x)$ as *logarithms to the base* b, denoted $\log_b x$, where b is any number greater than 1. Hence, with this new notation,

$$\frac{L(x)}{L(b)} = \log_b x.$$

However, $L(x)$ alone can be written $\log_e x$ and similarly $L(b)$ may be written as $\log_e b$. Thus,

$$\log_b x = \frac{L(x)}{L(b)} = \frac{\log_e x}{\log_e b},$$

and the change-of-base formula is established. And, of course $\log_e x$ is denoted $\ln x$.

With this established it is then an easy matter to discuss the properties of the inverse of the function $L(x)$, which I oh-so-cleverly call $E(x)$. Naturally, I lead the students to the fact that the inverse is the exponential function. But before we get there, I exploit the fact that $E(x)$ is the inverse of $L(x)$ in order to find the derivative of $E(x)$.

I begin by composing $\ln(x)$ with $E(x)$ in two ways. The first way is $\ln(E(x)) = x$. Recalling that $\ln(x) = \int_1^x \frac{1}{t}\, dt$, we also have that

$$\ln(E(x)) = \int_1^{E(x)} \frac{1}{t}\, dt.$$

Therefore,

$$\int_1^{E(x)} \frac{1}{t}\, dt = x.$$

Taking derivatives of both sides, we get

$$\frac{1}{E(x)} \cdot E'(x) = 1, \qquad \text{or} \qquad E'(x) = E(x).$$

This is the simplest way of actually proving that the derivative of the exponential function is itself. Only the development of the logarithm as an integral allows this. When we introduce exponentials and logarithms early in the AP Calculus course, we lose this mind-blowing result—instead, it becomes something only accessible through calculator tricks and numerical data. Which is a fine method if no other method is available; but there is, I think, a better method in this case.

Of course when it comes to exponentials, the traditional method of writing the inverse of the logarithm is by denoting it e^x. This is a fine notation—as far as it goes. However, when introduced using this notation, it becomes too easily confused in the minds of students with a power function. I know we have all seen the following ridiculous work in our students:

$$\frac{d}{dx}\left(e^{x^2}\right) = x^2 e^{x^2-1}. \qquad \text{(This is wrong!)}$$

How can we get our students to stop making this silly mistake? By never using this notation until later in the course. Instead

USE THE NOTATION $\exp(x)$ FOR THE EXPONENTIAL FUNCTION.

When students are asked to differentiate $f(x) = \cos\left(x^2\right)$, $g(x) = \ln\left(x^2\right)$ and $h(x) = \sin\left(x^2\right)$, the students do not use the power rule by mistake! They of course use the chain rule: $f'(x) = -2x\sin\left(x^2\right)$, $g'(x) = 2x(1/x^2) = 2/x$, and $h'(x) = 2x\cos\left(x^2\right)$. So why not use a notation for the exponential function which reinforces the fact that it is a function like these others? With this notation, a student will correctly obtain

$$\frac{d}{dx}\left(\exp\left(x^2\right)\right) = 2x\exp\left(x^2\right).$$

Like the logarithm function, the trigonometric functions, and the inverse trigonometric functions, we should use a notation that screams "function!"

What about the other exponential functions like 2^x and 3^x and the like? Well, since each of these can be expressed in terms of the function $\exp(x)$, like so—

$$2^x = \left(e^{\ln 2}\right)^x = e^{x\ln 2} = \exp\left(x\ln 2\right)$$

—one can avoid using any other exponential function but $\exp(x)$, just as one can avoid using any other logarithm function but $\ln x$.

Speaking of avoiding troublesome notations and functions, I have a bone to pick with the notation for inverse trigonometric functions. I can't stand it. That's why I

USE THE "ARC" NOTATION FOR INVERSE TRIG FUNCTIONS.

I cannot count how many times students see $\sin^{-1} x$ and think "$\csc x$." It is ingrained in their little heads that an exponent of negative one must mean reciprocal. I have tried to unteach that fact, but unteaching is really hard to do! So I have resorted to using "$\arcsin x$" for the inverse sine to avoid any confusion with the cosecant.

I much prefer this notation for another reason as well: this notation indicates what this function does. We know that we need an angle input for the sine function; that is obvious. But what do get out? When we write $\sin(\pi/6) = 1/2$, what exactly does that $1/2$ represent? It represents a length. Indeed, given a circle of radius 1, $\sin\theta = k$ says that the length of the chord determined by the angle 2θ is $2k$. In ancient times, the value of the length of the chord was used to denote the length of the arc subtended by the chord. That is, by constructing a chord of length $2k$, one then determines an arc, so the length of the chord became associated with the length of the arc. Therefore, the sine function takes an angle and from that one may determine the length of an arc. It makes sense that the inverse sine function should take a length of an arc and return the associated angle; hence "arcsine." I tell my students simply that sin(angle) = length and arcsin(length) = angle.

So not only does this notation resolve the confusion between $\sin^{-1} x$ and $\csc x$, this resolves most of the domain and range issues students have with these functions as well.

Those domain and range issues are tricky, but getting a handle on them is important! Most students ignore domain issues, particularly when a function is defined on an interval. This is especially important when finding extrema, which is why I

USE THE "CLOSED INTERVAL EXTREMA TEST."

I teach the two classic tests for extrema: the first derivative test and the second derivative test. (The second derivative test is a test for extrema, not inflection points—there is an "inflection point test" for that.) Most AP Calculus teachers are good with teaching their students these tests. Most are also good with simply evaluating the function at the critical points and at the endpoints when the function is defined on a closed interval. But the students do not so a good job of catching the closed interval problems. By teaching this as a separate test, students are keyed into noticing whether the function is defined for an interval or not.

I teach many things not mentioned in the AP Calculus AB course description, such as l'Hôpital's rule. I am careful about the "extra" stuff I teach. I feel anything extra must make sense to the students and there must be a reason for its inclusion. I also teach things not mentioned in the AP Calculus BC course description. In fact, I

TEACH THE ROOT TEST FOR CONVERGENCE.

The root test for infinite series says the following.

Theorem 1 (Root Test)**.** *For the series $\sum_{n=1}^{\infty} a_n$, define*

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $r < 1$, then $\sum a_n$ converges absolutely; if $r > 1$, then $\sum a_n$ diverges.

Any series which can be shown to converge by the ratio test also can be shown to converge by the root test. (In fact, both the ratio and root tests have the same conclusion.) As long as one establishes that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$,[1] then the root test is, in my opinion, easier to use than the ratio test. For example, most of us would use the ratio test to solve the following problem.

Problem 1. *Determine the convergence or divergence of the series*

$$\sum_{n=0}^{\infty} \frac{(n+1)^n}{n!n^n}.$$

The ratio test certainly works, but the root test determines the convergence more quickly:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(n+1)^n}{n!n^n}\right|} = \lim_{n\to\infty} \frac{n+1}{n\sqrt[n]{n!}} = 0$$

so that the series converges absolutely. The root test is easy to apply since it requires less algebraic manipulation than the ratio test. (I still teach the ratio test since there have been AP Exam problems

[1]This can be seen to diverge by splitting the expression up: $\sqrt[n]{1} \cdot \sqrt[n]{2} \cdot \sqrt[n]{3} \cdots \sqrt[n]{n}$. Each term converges to 1 from above, so this product consists of numbers each larger than 1, except for $\sqrt[n]{1} = 1$. Thus the product continues to grow as $n \to \infty$.

which specifically ask the students to use the ratio test.) Reducing algebraic manipulation is important for most of our students, since the algebra skills they do have are generally very weak. That is the motivation for why I

USE PARTIAL DERIVATIVES FOR IMPLICIT DIFFERENTIATION.

When confronted with an implicitly-defined curve, we AP Calculus teachers generally teach students to interpret each y as nothing more than representing an implicit function $y = f(x)$ so that the chain rule must be used on each y term. For instance, to find dy/dx for the curve $y^2 + x^3y^3 + x^2 - 8 = 0$ we find $2yy' + 3x^2y^3 + 3x^3y^2y' + 2x = 0$. Solving for y' we obtain

$$y' = \frac{-3x^2y^3 - 2x}{3x^3y^2 + 2y}.$$

The difficulty is that students forget to use the chain rule when differentiating y. This becomes more burdensome if the equation of the curve has many terms involving products or quotients of x and y. However, if we view the curve as defining a surface in space, then we may use partial derivatives to easily obtain dy/dx.

Suppose $F(x,y) = 0$ where F is a function of the two independent variables x and y. Then the total differential of this function is

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0.$$

(See [2].) Solving this differential for the ratio dy/dx gives

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

Let's use this technique on the implicit function $F(x,y) = y^2 + x^3y^3 + x^2 - 8$. We have

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{3x^2y^3 + 2x}{2y + 3x^3y^2},$$

which is the same derivative we found before, but with much less trouble (and no chain rule errors)!

I think it is interesting that a change in notation, or a change in method, or a change in definitions, can lead to deeper understanding and easier execution. I personally had such an epiphany about eight years ago when I realized that

EULER'S METHOD IS JUST TANGENT LINES.

While my class was reviewing for the AP Exam, I happened to have done on the board a problem about tangent lines and another problem on Euler's method. I saw my Euler's method equation

$$y_{k+1} = \left.\frac{dy}{dx}\right|_{(x_k,y_k)} \Delta x + y_k$$

and I rewrote it:

$$\left.\frac{dy}{dx}\right|_{(x_k,y_k)} \Delta x = y_{k+1} - y_k.$$

I teach my students to write tangent lines in point-slope form:

$$m(x - x_1) = y - y_1$$

and I noticed that Euler's method is just tangent lines. The slope at a point is clearly $m = dy/dx|_{(x_k,y_k)}$ and the change in x, Δx, is $x - x_1$. This completely changed how I teach Euler's method. I teach this as nothing more that an application of tangent lines instead of as a strict numerical procedure. This approach eliminates one more equation for a student to memorize.

Note that Euler's method is itself an approximation for the value of a function. This is also the use of a tangent line: it too is an approximation. Tangent lines are very useful approximators. In fact,

TAYLOR POLYNOMIALS ARE JUST AN EXTENSION OF THE TANGENT LINE IDEA.

I introduce the concept of "approximating polynomials" early in the course. Once tangent lines are used to approximate functions, it is natural to ask how to get a better approximation. One answer is to add a quadratic term. This would match the concavity of the function. So we should use something like the following, to approximate $f(x)$ centered at $x = a$:

$$P_2(x) = A + B(x-a) + C(x-a)^2.$$

How do we know what the coefficients A, B, and C are? Well, certainly the values of f and P_1 must be the same at $x = a$. Therefore, we need $f(a) = P_2(a) = A$. Also, we want P_2 and f to be tangent; thus, we need $f'(a) = P_2'(a)$. Computing $P_2'(x)$ we get $P_2'(x) = B + 2C(x-a)$ so that $P_2'(a) = B = f'(a)$. This means that so far we have

$$P_2(x) = f(a) + f'(a)(x-a) + C(x-a)^2$$

and all that remains is to determine C.

We added the quadratic term so the approximating polynomial matches the concavity; so we need the *concavity* of $P_2(x)$ to match that of $f(x)$ at $x = a$. This implies that we need $f''(a) = P_2''(a)$. Thus $P_2''(x) = 2C$ so then $P_2''(a) = 2C = f''(a)$; this implies $C = f''(a)/2$.

At this point, I define the "quadratic approximating polynomial" for f at $x = a$ to be

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

where a is the center.

And of course, it is natural to ask about a cubic approximation. In fact, we can determine such an approximation using previous methods. Here, we want to determine coefficients A, B, C, and D such that

$$P_3(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3$$

where, again, $x = a$ is the center. We will end up going through the same procedure as with the quadratic approximation to obtain A, B, and C. Thus, we know $A = f(a)$, $B = f'(a)$, and $C = f''(a)/2$. It only remains to determine D. Following the pattern established, it makes sense that we need the third derivatives of f and P_3 to be equal at $x = a$. Computing, we have

$$\begin{aligned} P_3'(x) &= B + 2C(x-a) + 3D(x-a)^2 \\ P_3''(x) &= 2C + 6D(x-a) \\ P_3'''(x) &= 6D \end{aligned}$$

so then $P_3'''(a) = f'''(a) = 6D$, or $D = f'''(a)/6$. And, we have a definition of a "cubic approximating polynomial" of a function f at $x = a$ as

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3$$

where a is the center.

Discussions of convergence or divergence are held until Taylor series and Taylor polynomials are introduced later. However, this approach sets up the idea of Taylor series and Taylor polynomials. The students already understand the use of Taylor polynomials as approximators and can use them as such. The general form of a Taylor polynomial also makes sense, as students have seen and understood why it has that form. Many modern calculus books do not use this approach, although some old ones do. And a few old books use this idea very early in the course. This leads to one thing I would encourage every calculus teacher to do:

READ OLD CALCULUS TEXTBOOKS.

Older calculus textbooks contain gems not contained in new textbooks. (Why is that? Do mathematical terms fall out of favor? Are certain symbols and notations just a passing fad? Are modern textbooks too concerned with providing support for technology to actually explain things well? I wish I knew.) The notation "$\exp(x)$" for the exponential function I found in an old calculus book (see [1] for example). Older calculus books were very concerned with curve-sketching—makes sense that they would be in the era before graphing calculators. However, notice that today in our calculus classes we do not spend any time at all sketching implicitly defined curves, even though our graphing calculators cannot graph them. So it would seem to me that knowing how to sketch implicitly-defined curves would continue to be useful to our students. Older calculus textbooks are filled with such ideas, while newer ones are not. I agree that technology should be used to graph these kinds of curves, but by disregarding all aspects of implicit curve-sketching, we lose some pertinent ideas for AP calculus. Allow me to describe one example of such a thing. Consider the following.

Problem 2. *Find all points on the curve $y^4-5y^2=x^4-4x^2$ at which the tangent lines are (a) vertical; (b) horizontal.*

Most of us would (quite rightly) find the derivative first. Let's use partial derivatives to find dy/dx. We write $F(x,y)=y^4-5y^2-x^4+4x^2=0$ and compute

$$\frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y}=-\frac{-4x^3+8x}{4y^3-10y}=\frac{2x^3-4x}{2y^3-5y}.$$

Then by the standard procedure we arrive at the points for which there are vertical tangents: $(\pm 2,0)$ and $(0,0)$; and for which there are horizontal tangents: $(0,\pm\sqrt{5/2})$ and $(0,0)$. However, in checking these points, we notice that the point $(0,0)$ actually makes the derivative take on the indeterminate form $0/0$. So how do we know that the tangents at the origin are really vertical (or horizontal)?

At this point, a newer textbook by a pair of eminent mathematicians instructs the students to use a calculator or computer. Using the calculator to find points on the curve close to the origin, they suggest calculating the average rate of change as an approximation to the slope of the tangent line. However, the old text by Love and Rainville [3] instructs students differently. That text includes the following definitions:

Definition. A *singular point* is a point on a curve at which the derivative becomes $0/0$. This indicates the presence of a *double point*—a point through which the curve passes twice. The tangent lines of the curve at a double point are found by considering only those terms of degree two in the equation and solving.

Since the terms of degree two are $-5y^2$ and $4x^2$, we ignore the other terms and solve $-5y^2 + 4x^2 = 0$ to get $y = \pm 2x/\sqrt{5}$; these are the tangent lines at the origin (which, by the way, proves that the tangents at the origin are neither vertical nor horizontal). This is a very nice idea![2] As to why this works, I will refer you to [3].

Another old textbook demonstrates how to apply the multivariable calculus method of optimization by Lagrange multipliers to our standard single-variable optimization problems (see [4]). The result is that a complex optimization problem turns into a simple of system of equations. These old textbooks are unique for other reasons, too. One of them being that there aren't too many problems. There seem to be just the right mix of routine and challenging problems for students to try—unlike modern textbooks. Which brings me to the last thing I learned:

Modern calculus textbooks have too many problems.

Open up any of the current favorite calculus textbooks and you will see something that should shock you: every section of every chapter has 70-100 problems. But my guess is that this doesn't shock you. This is probably something you are used to seeing in all calculus textbooks. But consider this: Is it possible for one student to do them all in a semester or two? Is it possible that any teacher, professor, or teaching assistant would grade them all even if a student did them? I do not think so. And the publishers of the textbooks don't think so either, or they wouldn't produce "teacher's guides" which suggest that students do "every third problem." If the publishers themselves produce guide books suggesting students are only to do every third problem, then why not eliminate two-thirds of the problems and cut the book down by 200 pages?

And this makes us lazy teachers. If the guidebook says to only assign every third problem, then that's what we do, isn't it? It's funny how we trust those guidebooks. Did we read every third problem to see if it was appropriate to gauge our students' understanding of what we taught them? Did we *do* every third problem in order to anticipate the struggles our students may face? More often than not, the answer is no—we simply make the students struggle with problems that may not have been appropriate and then wonder why they give up on doing homework as the semester drags on.

Of course, the reason books have so many problems is that the publishers feel that to make a profit the book must be marketable to as many high schools, colleges, and universities as possible. So the publishers hire "assistants," "accuracy checkers," "art program designers," and myriad others to "help" the author write the book, develop problems, write solution manuals, and produce guidebooks which advocate doing every third problem. The textbook becomes a book written by committee instead of expressing the educational and mathematical vision of the author, and so the book becomes like all other textbooks: a 1200-page calculus doorstop. Why do we continue to purchase such nonsense?

References

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[4] Robert Osserman, *Two-Dimensional Calculus*, Harcourt Brace and World, New York, 1968.

[2] This method only works if the double point is located at the origin. If the double point is located somewhere else, then one must translate the curve so that the double point lies on the origin. For instance, if the double point of the implicit curve $f(x,y) = 0$ is $(2,3)$, then the curve must be translated to the left 2 units and down 3 units, creating the curve $f(x+2,y+3) = 0$.

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