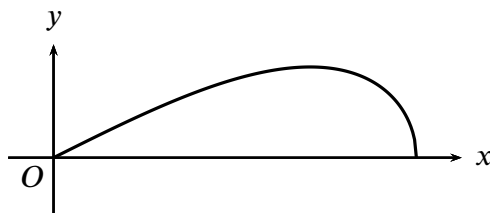


2 **Problem Overview**

3 The student is given a portion of the graph of the continuous function  $y = cx\sqrt{4 - x^2}$  (shown below) for  
4 some positive constant  $c$ . The function and the  $x$ -axis bound a region of the first quadrant. When this  
5 region is revolved around the  $x$ -axis, this resulting solid forms the basis of the design of child's spinning  
6 top.

7  
8 **Part a**

9 Students were asked to find the area of the region given that  $c = 6$ .

10 **Part b**

11 Students were told the derivative  $\frac{dy}{dx} = \frac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$  and that for a particular top, the largest cross-section  
12 has a radius of 1.2 inches. Students were asked to find the value of  $c$ .

13 **Part c**

14 Students were told that for a different top, the volume is  $2\pi$  cubic inches, and they were asked to find the  
15 value of  $c$  for this toy.

## 16 Comments on Student Responses and Scoring Guidelines

### 17 Part a

18 This problem involved the set up and evaluation of a definite integral to calculate the area of the region,  
19 using  $c = 6$ . There were three points available in this part. The first point was for presenting either  
20  $cx\sqrt{4-x^2}$  or  $6x\sqrt{4-x^2}$  as the integrand of a definite integral. Since this point was only for the integrand,  
21 the limits of integration did not have to be correct to earn this point. If the correct integrand was presented in  
22 an indefinite integral, the student could earn this point if the correct antiderivative was eventually evaluated  
23 using the correct limits.

24 Then students had to find the antiderivative. The correct antiderivative earned the second point. This could  
25 have been accomplished with the substitution  $u = 4-x^2$  or with  $u = x^2$ , or it could have been accomplished  
26 without any explicit substitution. The substitution the student used and the resulting substitution of  $dx$  was  
27 read for this antiderivative point. If the student used  $u = x^2$ , then  $du = 2x dx$  is correct, but if the  
28 student used  $u = 4-x^2$  but dropped the negative in  $du = -2x dx$ , the student did not earn this point. If  
29 the student entered into this process with a bad value of  $c$  (such as using  $2x\sqrt{4-x^2}$ , which would have  
30 cost the student the first point), we still read for the antiderivative point. The student needed to show an  
31 antiderivative of a function of the form  $Kx\sqrt{4-x^2}$  where  $K$  is the student's coefficient in the integrand.  
32 If the student used a  $u$ -substitution, the limits of integration should also be changed, but because this did  
33 not affect the antiderivative, errors with limits did not make the student ineligible for this point. Depending  
34 on the substitution the student used, Readers needed to see an antiderivative of the form

$$35 \quad \frac{K}{3}(4-x^2)^{3/2} \quad \text{or} \quad \frac{K}{3}u^{3/2}$$

36 with the correct use of the negative from any substitution the student used.

37 The third point was for the answer, 16, for which the student was eligible only if the student earned the  
38 antiderivative point. Any errors in limits, simplification of the antiderivative, the coefficient in part (a),  
39 arithmetic, or signs did not earn this point. If the student calculated the answer as  $-16$ , the student did not  
40 earn this point as this could only happen by an error with the limits of integration. (If the student realized  
41 that the area could not be negative and subsequently declared the area to be  $+16$ , the student still did not  
42 earn this point as the integral was not evaluated correctly.)

### 43 Part b

44 This problem could earn the student two points. The first point is earned by setting the given derivative  
45 equal to 0. This could be demonstrated by any of the following.

$$46 \quad \frac{dy}{dx} = 0, \quad \frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0, \quad c(4-2x^2) = 0, \quad \text{or} \quad 4-2x^2 = 0.$$

47 Obtaining the solution  $x = \sqrt{2}$  is important for subsequent calculation, but obtaining the solution was not  
48 necessary to be awarded this point.

49 The second point was earned only for the answer  $c = 0.6 = 3/5$  with supporting work. A student who used  $x = \sqrt{2}$   
50 used correctly to obtain  $c = 0.6$  earned the answer point, but not the first point.

### 51 **Part c**

52 Four points are available to the student in part (c). To earn the first point, the student had to present an  
53 integrand of the form  $K \left( x\sqrt{4-x^2} \right)^2$  in a definite integral. The limits of integration did not need to be  
54 correct to earn this point. Student mistakes in handling the constant  $c$  will render the student ineligible  
55 only for the answer point.

56 To earn the second point, the student must have the correct limits of integration ( $x = 0$  and  $x = 2$ ) and the  
57 constant  $\pi$  as part of a definite integral. A student presenting an incorrect integrand (which did not earn  
58 the first point) was still eligible for this point. The use of  $2\pi$ ,  $3\pi$ ,  $\pi^2$ , or anything other than  $\pi$  as a constant  
59 did not earn this point. If the student presented an indefinite integral with the correct constant  $\pi$ , this point  
60 could be earned by subsequent evaluation of the student's antiderivative at the correct limits.

61 A student who enters the problem by writing

62 
$$2 = \int_0^2 \left( cx\sqrt{4-x^2} \right)^2 dx$$

63 earns both the first and second points.

64 To earn the third point, the student had to present the correct antiderivative of a function of the form  
65  $K \left( x\sqrt{4-x^2} \right)^2$ , where  $K$  is the student's coefficient. Again, if the student made any errors in handling  
66 the constant  $c$  the student is eligible for this point, but not the answer point. This antiderivative must be of  
67 the form

68 
$$K \left( \frac{4}{3}x^3 - \frac{1}{5}x^5 \right).$$

69 Any errors in simplification of the antiderivative, bad coefficients from the integrand, or sign errors leaves  
70 the student eligible for this point, but not the answer point.

71 The fourth point is earned only for the answer  $c = \sqrt{15/32} = \sqrt{30}/8$ . This does not need to be simplified  
72 or rationalized to earn the point.

### 73 **Observations and Recommendations for Teachers**

74 (1) Students should read the problem. In part (a), some students misread the problem. Instead of using  
75  $c = 6$ , many students used 6 as the upper limit of integration. In part (b), some students clearly did not  
76 understand the information given nor what they were being asked to do. Students tried to integrate  $dy/dx$ ,  
77 or to plug in 1.2 into the derivative or the function, or attempted to set the derivative equal to 1.2 and solve.  
78 (Uncannily, the student who set  $dy/dx = 1.2$  and then set  $x = 0$  and solved for  $c$ , obtained  $c = 0.6$ . This  
79 did not earn either of the two points.) Reading the problem statement and understanding what is required  
80 to do is a skill that needs to be practiced with students.

(2) Students should consider their answers. In part (a), students using the incorrect limits of integration with a substitution obtained some interesting answers. For instance, consider the student who used the substitution  $u = 4 - x^2$  properly, and changed the limits correctly to  $u = 4$  and  $u = 0$ . The student writes, correctly,

$$\int_0^2 6x\sqrt{4-x^2} dx = \int_4^0 -3\sqrt{u} du = \int_0^4 3\sqrt{u} du = 2u^{3/2} \Big|_0^4.$$

But then the student forgets to change the limits of integration back:

$$2u^{3/2} \Big|_0^4 = 2(4-x^2)^{3/2} \Big|_0^4 = 2((-12)^{3/2} - 4^{3/2}).$$

Some students stopped here, declaring this as the answer, but others kept going, declaring the area is imaginary or that it does not exist (even though there is a graph on the paper which shows the region). Teachers should train students to consider the legitimacy of their answers as a signal that something might be incorrect.

(3) Students should read the problem. A remarkably large number of students appeared to believe that the function  $y = cx\sqrt{4-x^2}$  already described the area of the region. These students used no integrals of any kind in part (a) nor in part (c).

(4) Students should read the problem. A remarkably large number of students seemed to think that part (a) was a volume problem and set up a required volume integral, as asked for in part (c). Many of these students evaluated it correctly. However, this made the students ineligible for all the points in part (a). Indeed, if Readers saw the integrand squared in part (a), the student was ineligible for any of the three points in part (a). Likewise, if Readers saw the integrand *not* squared in part (c), the student was ineligible for any of the four points in part (c).

(5) Students should read the problem. Some students believed that the derivative in part (b) needed to be differentiated to solve the problem, and others thought it needed to be antiderivatified. (Strangely, almost no one who did this recognized that the antiderivative of  $dy/dx$  is  $y = cx\sqrt{4-x^2}$ .) Part (b) was a fairly straightforward algebra problem, but it required knowledge of calculus to know how to set-up the problem. This should have been an easy two points, but nearly 70% of papers scored by this Reader earned zero points on part (b).

(6) Students should know how to find antiderivatives. It is not often that a single free-response problem requires the student to find two antiderivatives, and these two on this problem are not overly complicated. In part (c), some students actually used integration by parts to integrate  $\int x^2(4-x^2) dx$  rather than simply multiply to get  $\int (4x^2 - x^4) dx$ . (Only one person who used parts did it correctly.) Of course, some students still believe that  $\sqrt{4-x^2}$  is the same as  $2-x$ . Teachers should spend time with students on substitutions, and using algebra to simplify integrands.

(7) Students should attend to precision. Perfect student work was marred by dropping negatives, bad arithmetic, disappearing exponents, or miswriting fractions. In part (a), more than once a student wrote  $u^{2/3}$  as the antiderivative of  $u^{1/2}$ . In part (c), students wrote the antiderivative of  $x^4$  as  $\frac{1}{5}x^4$ , and  $(cx\sqrt{4-x^2})^2$

116 was often expanded as  $cx^2(4 - x^2)$  instead of  $c^2x^2(4 - x^2)$ . Teachers should place an emphasis on such  
117 precision; these mistakes could be costly on the AP Exam (such as the lack of a squared integrand in part  
118 (c)).