2021 BC-5

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Problem Overview

The students were told that $y = f(x)$ is the solution to the differential equation

$$\frac{dy}{dx} = y \cdot (x \ln x) \tag{I}$$

with the initial condition $f(1) = 4$. The students were also told that $f''(1) = 4$.

Part a

Students were asked to write the second degree Taylor polynomial for f about $x = 1$. They were then asked to use this polynomial to approximate $f(2)$.

Part b

Students were asked to approximate $f(2)$ using Euler's method with two steps of equal size starting at $x = 1$.

Part c

Students were asked to find $y = f(x)$, the particular solution to the differential equation (I) with the inital condition $f(1) = 4$.

Comments on Student Responses and Scoring Guidelines

Part a: worth 2 points

The first point for part a was awarded for the correct polynomial equation. Students were not required to arithmetically simplify the coefficients but the polynomial had to be centered (written about) $x = 1$.
The second point was for the approximation. If the first point was earned, then this second point could be earned by just writing "6". No supporting arithmetic work was required. If the student did not earn the first point, the the student needed at least "$4 + 2(1)$" to earn the second point.

Part b: worth 2 points

Students needed two steps (with size 1/2) of Euler's method to earn the first point. The students could have at most one error to earn this point. If there was an error, then the second point which was for the answer, was lost.
The correct application of Euler's method required two steps of size $\Delta x = 1/2$, the use of the correct expression of $\frac{dy}{dx}$, and the use of the initial condition $f(1) = 4$.

The two steps of Euler's method could be explicit expressions or they could be presented in a table. In the presence of the correct answer, a table did not need to be labeled in order to earn both points. If an incorrect answer was presented, then the table must be labeled correctly to earn the first point.

x	y	$\Delta y = \frac{dy}{dx} \cdot \Delta x$ or $\Delta y = \frac{dy}{dx} \cdot (0.5)$
1	4	0
3/2	4	$3 \ln 3/2$
2	$4 + 3 \ln .5$	

Minimal example of correctly labeled table

If a student made an error in computing the approximation at $x = 3/2$ and used it in the second step, then this was not considered a second error. The minimal amount of response required to earn both points is

$$4 + 0.5 \cdot 4 \cdot \ln 1.5$$

Part c: worth 5 points

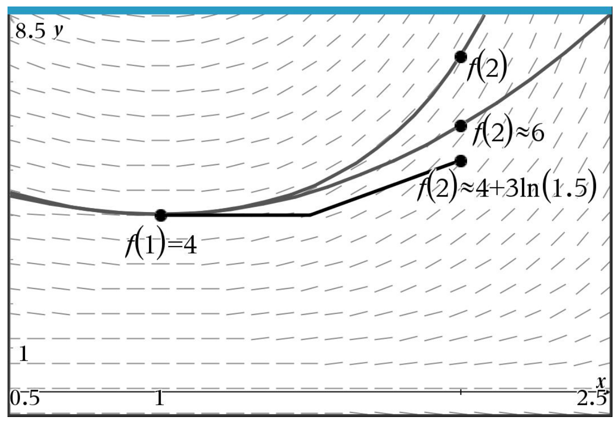
The first point for part c was for separating the variables. Separating the variables was the most important step, as a response without any attempt to do so earned none of the five points for this part. If students separated the variables incorrectly, but left one side correct, then they could earn the corresponding antiderivative point for that side.
There were two antiderivative points, one for each side. Students were required to compute $\int x \ln x dx$ and $\int \frac{dy}{y}$. The antiderivative of $\frac{1}{y}$ could be earned with either $\ln |y|$ or $\ln y$.
The fourth point was earned for the use of the initial condition in the presence of a constant of integration. If the student response did not contain a constant of integration, then the most the student could earn was the three points of the five available for this part. The student also needed to have earned at least one of the antiderivative points to receive the fourth point.
The fifth and final point is earned when the student solves for y. Students could only earn this point if they had earned all four of the previous points for this part.

Observations and Recommendations for Teachers

(1) Many AP Calculus BC students entered the exam very adept at tabular integration by parts but do not have much practice outside of problems better suited for a straight forward approach. This problem utilizes an important family of integrals of the form shown below.

$$\int x^n \ln x dx$$

(2) This problem has the student explore both the solution to a differential equation as well as two methods for approximating these solutions. Teachers should emphasize clearly to students the difference between an approximation and an actual solution. Many students reported their approximations for part (b) as being equal to the value of $f(2)$. The graph shows the difference between the two methods of approximation and the solution to the differential equation.



Approximation Versus Exact Value

(3) The ability of calculators to perform approximations to differential equations using Euler's Method has relegated problems such as part (b) to the non-calculator section of the exam. Students who are uncomfortable with arithmetic computations should practice writing these computations without simplification. The questions never require more than two steps and so the arithmetic expression does not become too long. An example for part (b) is shown below.

$$f(1.5) \approx 4 + 0(1.5 - 0)$$

$$f(2) \approx (4 + 0(1 - 5.0)) + 6 \ln 1.5(2 - 1.5)$$

(4) Most textbooks do not provide students with opportunities to solve differential equations which require both separation of variables and integration by parts. This is the first example I remember of it occurring in a Free Response Question. Teachers, who wish to offer students more chances to work with such problems outside their normal textbook, should consider *The AP Calculus Problem Book* by Chuck Garner.