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**Problem Overview:**

Students were given that a particle was moving in the xy-plane with position $(x(t), y(t))$. At $t = 0$, the position of the particle was $(-2, 5)$. The velocity vector for the particle was $\langle (t-1)e^{t^2}, \sin(t^{1.25}) \rangle$.

**Part a:**
Students were asked to find the speed and the acceleration vector of the particle at time $t = 1.2$.

**Part b:**
Students were asked to find the total distance traveled by the particle over the time interval $0 \le t \le 1.2$.

**Part c:**
Students were asked to find the coordinates of the point where the particle is farthest left for $t \ge 0$ and explain why there is no point at which the particle is farthest to the right for $t \ge 0$.

**Comments on student responses and scoring guidelines:**

**Part a** worth 2 points

For the first point, students had to apply the formula for speed at $t = 1.2$, presenting a correct answer to three decimal places along with supporting work: $\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271$ Supporting work is the appropriate radical expression for speed without any linkage errors. Writing the generic formula for speed was not sufficient.

For the second point, students had to present the acceleration vector at $t = 1.2$ to three decimal places with supporting work. $\langle x''(1.2), y''(1.2) \rangle = \langle 6.246, 0.405 \rangle$ *or* $\langle 6.247, 0.405 \rangle$ . Other vector notations were accepted as well as the coordinates listed separately as long as they were labeled.
Supporting work must indicate the derivatives being evaluated to get the acceleration vector. Many students wrote expressions for the components of the acceleration vector instead of the values.

Points were not awarded for answers without supporting work.

**Part b** worth 2 points

The first point was awarded for the correct integrand of any definite integral:
$\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Students must be careful when using the given expressions. Often, forgetting or misplacing a parenthesis resulted in an incorrect integrand.

In the presence of the correct limits, the second point was awarded for the correct answer.

$\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.009 \text{ } or \text{ } 1.010 \text{ } or \text{ } 1.01.$

Students were forgiven if missing the differential $dt$ or a misspelling $dx$ in the presence of the correct answer.

If the first point was not earned because of a copy error, the correct answer could earn the second point.

Unsupported answers earned neither point.

**Part c** worth 5 points

To find the coordinates of the point farthest left, it is necessary to find the critical point of $x(t)$. The first point is for setting $x'(t) = 0$. This equation is true for $t = 1$.

The second point is a global argument explaining the left most position at $t = 1$. A sign chart is not sufficient to earn this point. Students may verbalize the sign chart or the motion of the particle by saying that $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, thus the particle is farthest left at $t = 1$.

The third point is one of the coordinates of the left most position with support.

$$x(1) = x(0) + \int_0^1 x'(t)\, dt = -2.603 \text{ } or - 2.604$$

$$y(1) = y(0) + \int_0^1 y'(t)\, dt = 5.410 \text{ } or \text{ } 5.41$$

The use of $dx$ or missing $dt$ was forgiven unless an ambiguous integral was presented like $x(1) = \int_0^1 x'(t) - 2$. In this case it is unclear if $-2$ is a part of the integrand.

The fourth point is for the coordinates of the left most position with support. The coordinates do not need to be written as an ordered pair, as long as the coordinates are labeled. Vector notation for the coordinates was accepted.

$$(-2.603, 5.410) \text{ } or \text{ } (-2.604, 5.41)$$

The fifth point is for the explanation that there is not a farthest right point. This communication must verify that the particle moves past the initial position.

The particle is moving left for $0 < t < 1$ and then for $t > 1$ moving right. The key is to justify and communicate that the particle moves past the initial position of $x = -2$. One way to do this is to provide a position after $t = 1$ such as $x(2) = x(0) + \int_0^2 x'(t)\, dt > -2$ .

The fifth point can be earned in other ways.

For example, an argument noting that as $t$ approaches infinity, $x(t)$ approaches infinity, and $x$ approaches infinity is true.

However, stating as $t$ approaches infinity, the particle moves right is not sufficient to show that the particle moves right of the initial point.

Students must be careful not to misspeak by saying $x$ increases to infinity for $t > 0$ as this is not true for $0 < t < 1$.

**<u>Observations and recommendations for teachers:</u>**

(a) This was a calculator active question. In many cases students showed step by step calculations for the speed and often listed step by step calculations for acceleration leaving expressions with 1.2 inputted for $t$. While correct, it was often unnecessary and sometimes resulted in a loss of the point because of miswriting the expression. Supporting work does not require step by step calculations or commentary of what is entered into the calculator.

Some students took the derivative of the given velocity vector to calculate the speed and then used the second derivative of the velocity vector to find acceleration. This misconception often carried through the entire problem. It is recommended that careful attention is paid to the information provided in the stem of the problem and that concepts like position, velocity, acceleration are labeled with the expressions provided. This is the first time in many years, students had to transfer expressions from the question book to the answer book. Maybe this contributed to these types of errors?

For the vector answer, many students wrote a vector in terms of $t$, and not the vector at $t = 1.2$ Students need to read questions carefully.

The presentation of incorrect work had missing parentheses and lots of linkage errors. For example, students would start stating the speed formula and continue with a string of equal signs linking numeric expressions to a variable expression.

Sometimes students presented the magnitude of the acceleration vector instead of presenting the vector components.

(b) Some students used acceleration instead of velocity in calculating the total distance.
Some students used distance values and not expressions.
Too many students did not write the integral with the differential $dt$. In a more complicated problem this notation error could result is a loss of points due to linkage. Proper notation is important and should be emphasized.
Students need to remember that they may use the names of the expressions such as $x'(t)$ instead of the complete expression $(t-1)e^{t^2}$. There were many errors in presenting the correct integrand. These presentation errors transferred to calculator inputs. Not having the correct sets of parentheses or correct exponents results in erroneous calculations.

(c) Part c has 2 communication points and 3 skill points.
To determine the point at which the particle is farthest left, the absolute minimum of $x(t)$ is needed. The first skill point is awarded for setting $x'(t) = 0$.
Once $t = 1$ is identified as when the particle is farthest left, some students did not justify their answer. Teachers should use this problem to emphasize the general directions “your work will be scored on the correctness and completeness of your methods as well as your answers.” Any time

a sign chart is used, students should verbalize the conclusions of the sign chart. Even when the phase “explain why” is not present.

The second communication point is for the explanation as to why there is no point farthest to the right. Many students overlooked that the justification must verify that the particle moves past the initial position since the particle moves left of the initial position then moves right. The statement “the particle moves right without changing direction for $t > 1$ is ambiguous. One interpretation of that statement is moving right and approaching $x = -2$. Explaining position based upon the left and right motion of the particle is a skill that is worthy of practice and attention to the evaluation of the correctness of the words in classroom practice.

Finding the coordinates of the farthest left point requires calculus work. For those students who listed coordinates, they were generally correct with supporting work. On occasion, students described graphs on their graphing calculator. Students should be reminded that describing graphs in general is not support for calculus answers.