2021 AB/BC1 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given a table of five values of an increasing, differentiable function $f(r)$ for five values of r. The function f is measured in milligrams per square centimeter and r is measured in centimeters. The function f gives the density of bacteria in a circular petri dish at a distance r from the center of the dish. The table is shown below.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Part a:

Students were asked to estimate $f'(2.25)$ using data in the table. This value had to be interpreted in the context of the problem, using correct units.

Part b:

Students were told that the total mass of bacteria in the petri dish, in milligrams, could be computed using the integral expression $2\pi\int_0^4 rf(r)\,dr$ and to approximate this value with a right Riemann sum using the four subintervals indicated by the data in the table.

Part c:

Students were asked if their answer in part b was an underestimate or overestimate of the mass of bacteria in the petri dish and to explain their reasoning.

Part d:

The function $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$ over the interval $1 \le r \le 4$ was given as a model for the density of bacteria in the petri dish. Students were asked to find the value of k on the interval $1 < k < 4$ for which $g(k)$ is equal to the average value of $g(k)$ on the interval $1 \le r \le 4$.

Comments on student responses and scoring guidelines:

Part a worth 2 points

The first point was earned for work showing a quotient and at least one difference, using values from the table. No simplification was required, but any simplification needed to be correct. Responses are expected to use values from the table nearest to 2.25: in this case $\frac{f(2.5)-f(2)}{2.5-2}$. To be eligible for the second point, a non-zero estimate needed to be shown. For the second point, the interpretation required references to $r = 2.25$, density of the bacteria, increasing or changing by 8, and units of milligrams per square centimeter per centimeter. An erroneous estimate that was negative required “decreasing at a rate of $|f'(2.25)|$” or “changing by $f'(2.25)$” in the interpretation.

Part b: worth 2 points

The setup for the right Riemann sum in part b requires a sum of four products. With at most one error in the Riemann sum, a response earns the first but not the second point. A completely correct left Riemann sum earned one of the two points; this value for a left Riemann sum is 91π, indicating that correct values of r were also used in the computation. A completely correct right Riemann sum with a value of 80π (indicating that values of r had been omitted) earned one of the two points in this part of the problem.

Part c: worth 2 points

For the right Riemann approximation to be an overestimate, the function must be non-negative and increasing. The function f gives values of density which must be non-negative. Also, values of r are non-negative. These two facts did not need to be stated. To show that the function $rf(r)$ is increasing, a response needed to appeal to the derivative $\frac{d}{dx}\left(rf(r)\right) = rf'(r) + f(r)$. The computation of the derivative using the product rule earned the first point. The second point was earned for stating that the integrand is non-negative and increasing and therefore the right Riemann sum is an overestimate. Responses attempting to reason from Riemann sum results in part b earned no points.

Part d: worth 3 points

The first point was earned for the definite integral setup $\tfrac{1}{3}\int_1^4 g(r)\,dr = g_{\text{avg}}$ with or without $\frac{1}{3}$ or $\frac{1}{4-1}$. A response of $\tfrac{1}{3}\int_1^4 g(r)\,dr$ earned both the first and second points with or without the result 9.875795. A value presented had to be correct to three decimal places after the point, rounded or truncated or the third point could not be earned. The third point was earned for the value $k = 2.497$. A response presenting the value of -13.995 was in degree mode and did not earn the second point. This degree mode response was eligible for the third point with either $k = 2.5$ or $k = 2.499$.

Observations and recommendations for teachers:

(1) Unless otherwise specified, an estimate of $f'(c)$ given values of a differentiable function $f(x)$ in a table has required use on the AP Calculus Exam of $a < c$ and $c < b$ where *a* and *b* are the values in the table closest to *c*. Work needs to be shown indicating an arithmetic calculation of a quotient of differences. Values should be pulled from the table in showing this work. Interpretations of such an estimate always require correct units, reference to the point *c*, and wording correctly describing increasing or decreasing. It is better to use these words rather than simply "changing" in the interpretation. Sometimes on the AP Calculus Exam, the word "changing" is not sufficient. It is better to be as descriptive as possible.

(2) To show that a Riemann sum is being calculated, work must show arithmetic indicating a sum of products, each product being a function value multiplied by a width. There is no formulaic approach to this computation because the widths can vary. Since the integrand in part b of this question is $rf(r)$, values of r needed to be included in the product calculations.

(3) Neither complicated mathematics nor calculus are needed in order to illustrate when a Riemann sum gives an overestimate or underestimate. Four sketches of both right and left sums for both increasing and decreasing functions illustrate the only possibilities. A quick scratch work sketch, meeting the conditions of the given problem, can serve as a reminder to students. Even on a high stakes exam there is time to make this sketch. The increasing or decreasing nature of the function may have to be verified using the sign of the derivative of the function, unless already given in the question. This concept is both easy to teach and for students to practice.

(4) Unless specifically required in a part of the question (and that hasn't happened on the AP Calculus Exam in years except for a sketch through a given slope field), **a student-provided graph is treated as scratch work. It is ignored by readers and not valid for use in an explanation or justification. To explain increasing or decreasing, calculus must be used, meaning an appeal to the sign of the derivative.**

(5) The average value of a function $f(x)$ on an interval [*a*, *b*] is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$. This is *NOT* to be confused with the average rate of change of the function on the interval given by $\frac{f(b)-f(a)}{b-a}$. AB/BC1 is a calculator active question. Students are expected to be able to compute an estimate of the value of a definite integral and to solve an equation, providing an answer correct to three decimal places. It is apparent from scoring many questions that some students still need to be reminded to be in radian mode when working with trigonometric functions.