## Problem Overview:

Students were told that the curve of the function $y=f(x)$ was defined by $2 y^{2}-6=y \sin (x)$ for $y>0$.

## Part a:

Students were asked to show that $\frac{d y}{d x}=\frac{y \cos (x)}{4 y-\sin (x)}$.

## Part b:

Students had to write an equation of the line tangent to this curve at the point $(0, \sqrt{3})$.

## Part c:

For $0 \leq x \leq \pi$ and $y>0$ students had to find the coordinates of the point where a line tangent to the curve is horizontal.

## Part d:

Students had to determine whether $f$ has a relative minimum, relative maximum or neither at the point found in part (c) and to justify this answer.

## Comments on student responses and scoring guidelines:

Part a worth 2 points
The first point was earned for a correct implicit differentiation of both sides of the given equation. If the first point was not earned, the second point could not be earned. The second point was for demonstrating that $\frac{d y}{d x}=\frac{y \cos (x)}{4 y-\sin (x)}$. This point could also be earned for work that resulted in $\frac{d y}{d x}(4 y-\sin (x))=y \cos (x)$ (and contained no subsequent work with any errors).

## Part b: worth 1 point

This point was earned for a correct tangent line equation, and no work had to be shown. Thus such answers as $y-\sqrt{3}=\frac{1}{4} x$ and $y=\frac{1}{4} x+\sqrt{3}$ earned the point. The slope did not have to be either calculated or simplified. So, using a slope of $\frac{\sqrt{3} \cos (0)}{4 \sqrt{3}-\sin (0)}$ was acceptable.

Part c: worth 3 points
The first point was earned for setting $\frac{d y}{d x}=0$. This could also be shown as $\frac{y \cos (x)}{4 y-\sin (x)}=0$ or as simply as $\cos (x)=0$. The second point was earned for $x=\frac{\pi}{2}$. If any other values of $x$ were given, students had to make a final commitment to $x=\frac{\pi}{2}$. Any values of $y$, correct or otherwise, were ignored in awarding the second point. The third point was earned for showing with some work that $y=2$. If additional values of $y$ were shown, students had to make a final commitment to $y=2$. Showing an ordered pair for this point was not required. Students entering this part of the problem with $x=\frac{\pi}{2}$, but showing no evidence of considering $\frac{d y}{d x}=0$, were eligible for the second and third points but could not earn the first.

## Part d: worth 3 points

An attempt at calculating $\frac{d^{2} y}{d x^{2}}$ using either a quotient or product rule earned the first point. Evidence of a quotient rule could be shown, for example, with a square in a denominator and a difference of terms involving derivatives in the numerator. The second point was earned for calculating the value of $\frac{d^{2} y}{d x^{2}}$ at the point $\left(\frac{\pi}{2}, 2\right)$ (or at $\left(\frac{\pi}{2}, k\right), k>0$ by importing an incorrect $k$ from part (c)). A correct answer and justification earned the third point. This had to include $\frac{d y}{d x}=0$ and the fact that $\frac{d^{2} y}{d x^{2}}<0$.

An alternate solution was also acceptable. Analyzing $\frac{y \cos (x)}{4 y-\sin (x)}$, the first point is earned for considering the sign of $4 y-\sin (x)$, demonstrated in such ways as stating that $4 y-\sin (x) \neq 0$. The second and third points can be earned without earning this first point. Stating that $\frac{d y}{d x}$ or $\cos (x)$ changes from + to - at the point where $x=\frac{\pi}{2}$ earns the second point using this method. The third point is for the conclusion that there is a relative maximum at $x=\frac{\pi}{2}$.

## Observations and recommendations for teachers:

(1) The basic calculation needed in part a is differentiating both sides of the equation in $x$ and $y$ implicitly. Grouping terms involving $\frac{d y}{d x}$ on one side of the resulting equation is the most direct approach to finding an expression for $\frac{d y}{d x}$. This involves some "symbolic algebra" manipulations, and students should practice this. While such grouping was sufficient in order to earn both points, it should be noted that the explicit expression for $\frac{d y}{d x}$ could possibly be required on future exams.
(2) While work in part $b$ was not required to be shown in computing the slope of this tangent line, that may not always be the case on the exam. It is not a bad idea to write out the arithmetic involved rather than computing something simple in one's head. Note that "an" equation of a tangent line can assume several forms. Solving explicitly for $y$ is not required for a problem worded in this manner
(3) In searching for relative extrema, it is almost always the case on the exam that consideration of $\frac{d y}{d x}=0$ must be shown in some manner for one point. Part c requires this. Entering such work with setting the numerator in the expression for $\frac{d y}{d x}$ earns a point. Students often display other values of $x$ which may not be in the given domain. It is important to exclude these and commit to the one(s) in the domain. In part c it was required to find the coordinates of the point, which means a value of $y$ to correspond to any chosen value(s) of $x$. Finding "the coordinates of a point" does not imply that an ordered pair be presented.
(4) Part c requires solving a quadratic equation in order to find the value of $y$. While explicitly testing precalculus skills on the exam has not happened since 1998, it is important to note that these skills are assumed. Students need to be presented with problems requiring such skills during their AP Calculus courses.
(5) Part d required a determination of the nature of the extremum, if there is one, at the point found in part c . Two basic methods for approaching this are: 1. Use the sign of the second derivative and apply the second derivative test or 2 . Examine the sign of the first derivative for change at the point in question. It is essential in applying the second derivative test that the first derivative be zero (and this should be stated in student work). The fact that the second derivative test fails if the second derivative is 0 was not relevant to part $d$ since there was a relative extremum at the point. An implicitly calculated derivative is not always easy to use in checking for a sign change in the first derivative test. In this part $d$ it was because the denominator is always positive. Students need to explain why only using the numerator to look for a sign change in the first derivative will work. In other words, all parts of a first derivative expression need to be examined using this analysis.

