2021 AB2 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given that particles P and Q were moving along the *x*-axis. At time, $t = 0$, P is at position $x = 5$. At time $t = 0$, Q is at position $x = 10$. The velocities of the particles are given as $v_P(t) = \sin\left(t^{1.5}\right)$ and $v_Q(t) = (t - 1.8)\bullet 1.25^t$.

Part a:

Students were asked to find the positions of both particles at time $t = 1$.

Part b:

Students were asked if particles P and Q were moving toward or away from each other at time $t = 1$ and to explain their reasoning.

Part c:

Students were asked to find the acceleration of particle Q at time $t = 1$. Further, along with the request for an explanation of reasoning, students were asked if the speed of particle Q was increasing or decreasing at time $t = 1$.

Part d:

Students were asked for the total distance traveled by particle P over the time interval $0 \le t \le \pi$.

Comments on student responses and scoring guidelines:

Part a worth 3 points

Initial positions of each particle were given at time $t = 0$. Students had to add this initial position to the displacement of the particle over the time interval $0 \le t \le 1$. The first point was awarded for the presentation in the work of either $\int_0^1 v_P(t)\,dt$ or $\int_0^1 v_Q(t)\,dt$, the setup for computing displacement. The second point was earned for presenting the position of one particle with work showing the addition of the initial position. The third point was earned by doing the same for the other particle. Use of *dx* rather than *dt* was forgiven. A missing *dt* was ignored unless in the ambiguous form $\int_0^1 v_P(t) + 5 = 5.371$ (or 5.370 or 5.37). In this presentation, it is unclear whether or not the number 5 is part of the integrand. Students doing this for both particles earned only the first point, even in the presence of two correct positions.

Part b: worth 2 points

Relative positions of the particles at time $t=1$ were part of the argument needed to explain why the particles were moving toward each other. In addition, directions of the particles' motions at $t=1$ were an important part of earning points in this part of the question. Many students began with $v_P(1)=0.841471$ (which needed to be correct to the number of digits presented or the second point could not be earned) and $v_Q(1)=-1$ (which needed to be correct or the second point could not be earned). These values were not required for this argument to earn points. No points were earned for these presentations as the first point was for the direction of one of the particles. This required an appeal to the sign of one of the appropriate velocities and usually the words "right" or "left." Responses not earning this point were eligible for the second point for a response such as "P is left of Q and moving toward Q because $v_P(1)>0$ and Q is right of P moving toward P because $v_Q(1)<0$." Some responses showed all this work using only $v_P(t)$ and $v_Q(t)$. Responses, no matter how correct they seemed to be, earned no points in part b if there were no reference to $t=1$. This scoring was partially because a number of responses tried to explain direction by referencing the initial positions as in "P is moving right because at $t=0$ P is at 5 and at $t=1$ P is at 5.371." Part b asks about what is happening at $t=1$, and references to the initial positions had no bearing on this explanation.

Part c: worth 2 points

This part of the question is about particle Q. The acceleration at time $t=1$ is 1.026856 and either 1.026 or 1.027 were acceptable for the first point. Work shown had to indicate that this value came from the derivative of v_Q at $t=1$. A response showing only $a_Q(1)=1.027$ did not earn the first point because of no connection to v'. The second point was earned for reporting that the speed was decreasing and referencing the different signs of $v_Q(1)$ and $a_Q(1)$. An erroneous value of $a_Q(1)$ not earning the first point could earn this second point if that value were positive. A response saying that the speed was slowing down did not earn the second point because it is the particle Q that is slowing down, not the speed which was the focus of this part of the question. As in part b, if there were no reference to time at $t=1$, no points were earned.

Part d: worth 2 points

This part of the question is about particle P. In order to calculate the total distance P traveled in the time interval $0 \le t \le \pi$, a response had to show $\int_0^{\pi} |v_P(t)|\,dt$ in order to earn the first point. The answer correct to three decimal places after the point is 1.931, and this earned the second point. Students could also calculate $\int_0^{\pi} |v_P(t)|\,dt$ using $\int_0^{2.145} v_P(t)\,dt - \int_{2.145}^{\pi} v_P(t)\,dt$ if it was made clear that the 2.145 came from solving the equation $v_P(t)=0$. The 2.145 did not have to be reported to three decimal place accuracy, but some students who did early rounding presented an incorrect answer, possibly because of not having enough correct digits in their calculators. Variations of poor presentations such as $\int_0^{1} |v_P(t)|\,dt$ or $\int_0^{\pi} v_P(t)\,dt$ or $\int_0^{\pi} v_Q(t)\,dt$ required scoring notes or explanations to readers describing when a response could earn one of the two points in part d.

Observations and recommendations for teachers:

(1) The basic fact being applied in part a is the addition of displacement to an initial condition. Some students were clearly not aware of this concept, in some cases showing no work with any integrals. Those who were aware were sometimes "creative" in their notation to the point that the responses were difficult to read. For example, $X_P(t) = 5 + \int_0^t v_P(t)\,dt = 5 + .370$ or $X_P(t) = 5 + \int_0^1 v_P(1)\,dt = 5 + .370$. Too much mishandling of the *t* or the 1 did not earn both points.

(2) The simultaneous use of $v_P(t)$ and dx and even leaving off the differential were forgiven in the scoring. But the differential plays an increasingly important role in the integral setup as students move on in their studies to multivariate calculus. It's important to emphasize proper notation at this early stage of learning calculus. The difference between $\int_0^1 v_P(t) + 5 = 5.371$ and $5 + \int_0^1 v_P(t) = 5 + .371$ is too significant to ignore. As mentioned above, the first setup would only earn the first point for the definite integral, and one could consider that generous since it is unclear whether the integrand is $v_P(t)$ or $v_P(t) + 5$. Students in classrooms should always be required to show correct use of differentials in their written work.

(3) Students found the argument in part b difficult to complete. "At $t = 1$" requires a focus on the relative positions of the particles as well as the directions in which they are moving "at $t = 1$." In order to complete a sufficient argument, directions of motions need to be tied to signs of appropriate velocities at $t = 1$. This part of the question is not the most elementary application of concepts related to particle motion on an axis, but should be presented to students during study of this type of motion. The AP Calculus exam increasingly requires verbal explanations with justifications appealing to appropriate calculus computations (as well as the hypotheses involved when applying a theorem). Words such as "explain" and "justify" *require* verbal explanations.

(4) Part c asks a question that requires a comparison of the signs of acceleration and velocity at one particular time. This could possibly be thought of as the effect of the acceleration on the velocity. This effect can be determined by comparing the signs of velocity and acceleration at that time, in this case $t = 1$. It is crucial to mention the different or opposite signs and not use what might be referred to as "teaching terminology" by saying something like "the acceleration has an opposing effect on the direction of the velocity." One way to communicate that velocity and acceleration have different signs is $v_Q(1) \bullet a_Q(1) < 0$. This is a basic idea to analyze when studying particle motion and should be studied in an AP Calculus class.

(5) **Students must read the questions carefully.** Part a and b involved both particles P and Q. Part c involved particle Q. Part d switched focus to particle P. In part d, total distance travelled is computed, and that is a different idea than in part a. Also in part d, the time interval changed from $0 \le t \le 1$ to $0 \le t \le \pi$. To calculate total distance travelled requires the integral of the speed over the time interval $0 \le t \le \pi$. Since

speed is non-negative, any subintervals of $0 \le t \le \pi$ in which $v_P < 0$ can be dealt with either by using $|v_P(t)|$ as the integrand over the entire interval $0 \le t \le \pi$, or by separately integrating and negating results over subintervals where $v_P < 0$ and combining these distances with those for which $v_P \ge 0$. This latter method can be cumbersome if $v_P < 0$ over several subintervals. If calculator use is allowed, the preferred method is to use $|v_P(t)|$ as the integrand in order to find total distance travelled.