2019 AB6 Marshall Ransom, Georgia Southern University

Problem Overview:

Functions f, g and h are twice differentiable functions with $f(2)=g(2)=4$. The line $y=4+\frac{2}{3}(x-2)$ is tangent to both the graph of g at $x=2$ and the graph of h at $x=2$.

Part a:

Students were asked to find $h'(2)$.

Part b:

The function a is given by $a(x)=3x^2h(x)$. Students were asked to write an expression for $a'(x)$ and also to find $a'(2)$.

Part c:

It is stated that the function h satisfies $h(x)=\dfrac{x^2-4}{1-\left(f(x)\right)^3}$ for $x\neq 2$. It is known that $\lim\limits_{x\to 2}h(x)$ can be evaluated using L'Hopital's rule. Students were asked to use $\lim\limits_{x\to 2}h(x)$ to find $f(2)$ and $f'(2)$.

Part d:

It is known that $g(x)\le h(x)$ for $1<x<3$ and that k is a function satisfying $g(x)\le k(x)\le h(x)$ for $1<x<3$. Students were asked if k is continuous at $x=2$ and to justify their answers.

Comments on student responses and scoring guidelines:

Part a:

This part was worth one point only and only for the correct answer. There is no work to show and the answer did not have to be attached to mathematical notation as in $h'(2)=\frac{2}{3}$. Students reporting incorrect answers were eligible to use that in other parts of the problem. The most commonly reported incorrect answer was 4. Many students did show work which could usually be ignored by readers. Very rarely, student work was shown to calculate the correct answer in a manner that was incorrect resulting in no point for part (a).

Part b: worth 3 points

Calculating the derivative of this function requires the product rule. The first point was for showing the form of a product rule calculation and required four things: a sum of two terms, h in one term, h' in the other, and multiplication of each term by a polynomial (which could be a constant). If this was shown, students earned the first point. If students did not earn this first point because of committing common product rule errors given by $a'(x) = 6xh'(x)$ or $a'(x) = 3x^2h'(x)$ or any other error that did not meet the four criteria, they were not going to earn the second and third points. Special case: a number of students substituted $y = 4 + \frac{2}{3}(x-2)$ for h before calculating the derivative. Such students were not awarded the first and second points, but were eligible for the third point for a correct value of $a'(2)$. Another example: $a'(x) = 6x^2h(x) + 3x^3h'(x) \rightarrow a'(2) = 112$ would earn the first point for correct product rule form and the third point for an $a'(2)$ answer consistent with that form, but would not earn the second point for a correct expression of $a'(x)$. A few students expressed the derivative numerically perhaps as $36 \cdot 4 + 24 \cdot \frac{2}{3}$ and could only earn the third point. It is most wise to express a derivative using function notation and the variable on a test requiring work to be shown.

Part c: worth 4 points

As might be expected, students launched into the work at different places and readers needed to search for work that earned points. The first point was for finding a connection between $\lim\limits_{x\to 2} h(x)$ and the number 4. Some intervening work may have contained errors, but this connection ultimately shown earned the first point. Since L'Hopital's rule can be used to calculate $\lim\limits_{x\to 2} h(x)$ and $\lim\limits_{x\to 2}\left(x^2-4\right) = 0$, students needed to show that they were working with $1-\left(f(x)\right)^3 = 0$ in order to arrive at $f(2) = 1$. This earned the second point. (See **Observations and recommendations for teachers #4** below). The third point was for applying L'Hopital's rule. Student work needed to show three things: limit notation, a quotient, and convincing evidence of the attempt to calculate derivatives of both the numerator and denominator. Forms such as the correct $\lim\limits_{x\to 2} \dfrac{2x}{-3\left(f'(x)\right)^2 f(x)}$ or the incorrect $\lim\limits_{x\to 2} \dfrac{2x-4}{1-3\left(f'(x)\right) f(x)}$ earned this point. Using $= \dfrac{0}{0}$ attached to any work would not earn the third point, but such students were still eligible for the fourth point. Only the correct $-\frac{1}{3}$ was accepted for the award of the fourth point. The eligibility for this fourth point was earning the third point. The only exceptions to that eligibility for the fourth point were if the third point were denied solely because of the lack of limit notation or the connection $= \dfrac{0}{0}$.

Part d:

To earn just one point, students had to show/reference five things: "yes" for the answer; both h and g are continuous; $\lim\limits_{x\to 2} h(x) = \lim\limits_{x\to 2} g(x) = 4$; $\lim\limits_{x\to 2} k(x) = 4$; and $k(2) = 4$ because $h(2)$ and $g(2)$ both equal 4. If the Sandwich Theorem was named, some colloquialisms such as "Pinch Theorem" or "Squeeze Theorem" were acceptable, but no name was required. Specifically naming an incorrect theorem such as Rolle's or IVT disqualified the student from this point. Readers did not have to read much in part d. While there was lots of writing, a quick look could often easily discern that one of the five criteria was missing. I personally read hundreds of student responses to this question and only saw one that earned the point in part d.

Observations and recommendations for teachers:

(1) The basic fact being applied in part (a) is that a line tangent to the graph of a function h at a point where $x = a$ has a slope at that point equal to the derivative of h at that point. But the expression for the tangent line function is not equivalent to the function. A number of students substituted the expression for the tangent line function for h before computing the derivative. This will, of course, have the correct derivative of h at that point, but not necessarily anywhere else. More importantly, the tangent line is not h, meaning that the student after substituting is not working with the given function.

(2) The product and quotient rules are very basic computational skills. Far too many students reported a single term when calculating $a'(x)$ in part (b), indicating a lack of basic computational knowledge. The exam does require such computational facility, as well as for basic antiderivatives and using *u*-substitution.

(3) Students should read the question carefully, both before and after work. In part (b), both $a'(x)$ and $a'(2)$ were requested. While strong students can calculate quickly toward a value of $a'(2)$, leaving out work on the paper showing $a'(x)$ means that they were not answering part of the given question. Messages: substitute values for x after calculating an expression for a derivative (or antiderivative); and when done with work, read the question again to be certain that all that is requested has been addressed.

(4) Technically it is the $\lim\limits_{x \to 2} 1 - \left(f(x)\right)^3$ that equals 0. But in order to calculate $f(2)$ an equation can be used. Some students did show $1 - \left(f(x)\right)^3 = 0$ and concluded that $f(x) = 1$. The value of 0 is only guaranteed to be correct at the point where $x = 2$. Such students did not earn this point in part c.

(5) L'Hopital's rule on the exam has for the last few years has only involved the form $\frac{0}{0}$. But it is *never* correct to report that a limit $= \frac{0}{0}$. A limit is a value and $\frac{0}{0}$ is an indeterminate *form*, not a value. This will always cost the student an opportunity to earn at least one point somewhere when trying to apply L'Hopital's rule. Also potentially penalizing the student is poor notation linked with an equal sign. For example, after differentiating numerator and denominator, writing $\lim\limits_{x \to 2} h(x) = \dfrac{2x}{-3\left(f(x)\right)^2 f'(x)}$ is not correct because limit notation needs to be attached to the expression $\dfrac{2x}{-3\left(f(x)\right)^2 f'(x)}$. Limit notation does not "go away" until values are being calculated.

(6) The problem in part (d) is very difficult simply because there are so many things to show. This would make for a good class discussion. First, notice that this is a "Yes or No" question that needs to be answered. Second, working with the sandwich theorem requires limits of the functions on the left and right sides of the "sandwich". Thus limit notation must be shown. Third, this is about continuity which exists when the limit is the same as the value of the function. The value of the function must be verified as well as the limit and then these must be connected. Fourth, values for any function that come from a given limit value only exist if that function is continuous at that point. "Justify your answer" is a request for including more information such as the fact that a function is continuous (usually because it is differentiable) in order to state that the value of the function at a point is the same as the value of the limit at that point.