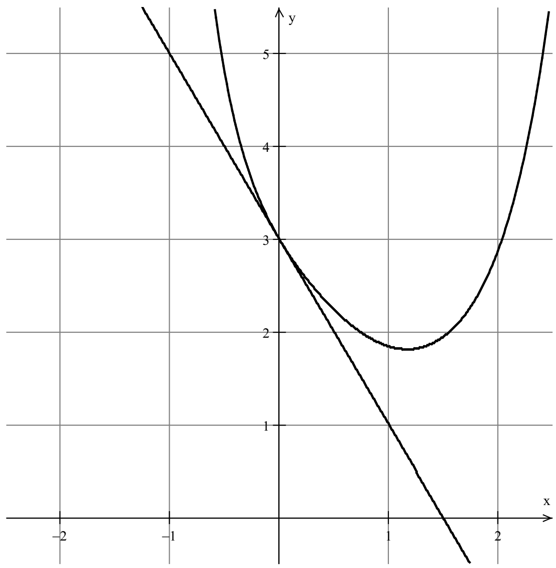
2019 BC-6 Dennis Wilson, Landmark Christian School

Problem Overview:



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

Students were given the graph of a function f along with the line tangent to the graph of f at $x = 0$. Students were also informed that the function f has derivatives of all orders and selected values for these derivatives at $x = 0$ were given in a table.

Part a:

Students were asked to write the third-degree Taylor polynomial for f about $x = 0$.

Part b:

Students were asked to write the first three terms of the Maclaurin series for e^x. Students were then asked to write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

Part c:

Students were told that the function h was defined as $h(x) = \int_0^x f(t)\,dt$ and asked to use the answer from part (a) to find an approximation for $h(1)$.

Part d:

Students were told that the Maclaurin series for h converges to $h(x)$ for all real numbers x. Students were also told that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Students were then asked to show that the approximation found in part (c) differed from $h(1)$ by at most 0.45 using the alternating series error bound.

General scoring guidelines for the problems:

Part a: **(2 points)**

In this part, student work was read for a correct third-degree polynomial. The first point came for any two of the four correct terms. The second point was earned for the other two correct terms. This point was lost however if students treated their answer as a series instead of a polynomial by including a tail $(+\cdots)$. If students lost the point for adding the tail in this part, they were inoculated for this error in part (b). Throughout this question though, readers were allowed to read the equals sign $(=)$ as approximately equals $(\approx)$. The second point was also lost if students presented any terms beyond the cubic term.

Students who entered this question performed well on this part. Some students struggled with determining $f'(0)$ from the given graphs. A few forgot to divide by the respective factorial for each term.

Part b: **(2 points)**

Students earned the first point for the first three nonzero terms for the Maclaurin polynomial of e^x. Students could give these terms as part of a polynomial, a series, or simply list them. These terms could not appear only in the student work leading to the Taylor polynomials for $e^x f(x)$. If these three terms were not by themselves or explicitly linked to e^x, then the student did not earn the first point.

The second point was earned for the correct second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$. The product of the two polynomials must be performed and the second-degree polynomial returned with terms collected. If students presented these terms with a tail $(+\cdots)$ or higher degree terms, then the second point was lost unless previously inoculated in part (a).

Students could still earn this second point even if they did not earn the points in part (a). If students imported a second-degree polynomial with three terms from part (a), they were still eligible in part (b). Students could also use incorrect terms for e^x in earning the second point as long as their incorrect terms included a constant term, a linear term, and a quadratic term.

Many students who performed well on other parts of this question struggled with part (b). Some students did not know how to properly multiply the terms for $f(x)$ and e^x. Other students performed the multiplication but failed to collect the coefficients for each respective power of x. Others did not attempt the multiplication of the two polynomials, instead calculating the coefficients individually by differentiating $e^x f(x)$ the necessary number of times. While some students were successful using this method, it consumed much more time than the alternative method.

Part c: **(2 points)**

Students earned the first point for the correct antiderivative of the polynomial in part (a). The second point was earned for the numerical answer. If students made only one error in the antidifferentiation, they could still earn the antidifferentiation point. Students could import an incorrect third-degree polynomial from part (a) and earn both points if the imported polynomial had at least two terms and the definite integral was performed correctly. If student made an error in antidifferentiation with an import, then the reader could read with the student only if the imported polynomial had four terms.

Many students substituted the values of 0 and 1 into $f(x)$ without calculating the antiderivative. It was unclear the reason students failed to perform this step. Other students though differentiated $f(x)$ in an attempt to find $h(x)$. This seemed to be a mental slip in the relationship between $f(x)$ and $h(x)$.

<u>Part d:</u> (3 points)

The first point in this part was earned for using the fourth-degree term of the Taylor polynomial for f about $x = 0$. To earn this point, students needed a term from a power series with evidence of anti-differentiation. This term could appear in a polynomial or a series. Explicit use of an integral sign was not necessary for this point. Any error in anti-differentiation would come off the second point.

The second point came for evaluating the fifth-degree term for h at $x = 1$. The student needed to commit to this term. If this term was part of a sum, then the point would not be earned. Linkage errors would come off this point. Linkage errors are when a student equates two expressions which are not equivalent. Students frequently use an equal sign as punctuation in their work. The Chief Reader Stephen Davis frequently says that "Equals is a verb, not punctuation" when referring to this mistake.

Two things were required by the student work to earn the third and final point of this part. First the student's solution needed some explicit reference to the error. This could be as simple as the word error, an algebraic expression for the difference, or a verbal description. The second thing required was a connection to the value given in the problem, that is 0.45. The decimal representation of this value or the fraction $\frac{45}{100}$ were necessary. Connections with any other numerical representations of this value could not earn this point. The reader could read less than ($<$) as ($\leq$) in this part.

Many students were unable to connect this part of the question with the necessary information from the stem to find the first omitted term. Other students used the first omitted term of the Maclaurin polynomial of $f(x)$ instead of that for $h(x)$.

Some students performed the calculus correctly but failed to earn the final point because they chose to use mathematical notation which was incorrect to answer the problem. This usually materialized as student attempted to make use notation such as $R_4(1) \leq |h(1) - P_4(1)|$. Without defining these terms, the reader could not award credit to the student for making a reference to error. Some students attempted to show that the alternating series error bound was applicable, even though the problem explicitly stated the conditions for it and instructed students to use it. A large number of students even tried to use Lagrange Error.

Observations and recommendations for teachers:

(1) Many students struggled with the simple act of dividing a fraction by a factorial. This led to students with good calculus work losing points for poor arithmetic. It cannot be emphasized more that both arithmetic and algebraic simplification are unnecessary. Students should perform only the calculus necessary to answer each question. Any additional mathematical computations simply use times better spent elsewhere and open the student for opportunity to lose points already earned.

(2) The Course and Exam Description specifically names four functions when referencing Taylor and Maclaurin Series (LIM-8.F.1 and LIM-8.F.2). These four are $\frac{1}{1-x}$, e^x, $\sin x$, and $\cos x$. Students should be required to commit the first four nonzero terms and the general terms for the Maclaurin series of these functions.

(3) Constructing Maclaurin Series from the previously mentioned series is an important skill for students. While substitution, differentiation, and antidifferentiation are the common operations, students should get practice computing the product of series. One useful exercise is to have students multiply and find the Maclaurin series for $\sin x \cos x$ and have them compare it to the Maclaurin series of $\sin 2x$ obtained using substitution.

(4) Students are frequently asked to determine a Taylor polynomial by antidifferentiating a polynomial found in a previous part. Students should be encouraged to provide an answer to these previous parts even if they do not know the proper method. With the guidelines given to readers for imports, students then have an opportunity for points on later parts.

(5) Many students, after using good calculus work, failed to earn the third and final point on part (d). If the students had simply restated the question as an answer, this point would have awarded. With the appropriate work to earn the first two points, the student could simply have stated "Therefore the approximation in part (c) differs from $h(1)$ by at most 0.45" and earned the last point. Students should be encouraged to answer the question that was asked. By rephrasing the question as an answer, student can often make a connection from their mathematical work to earning the points for it.

(6) The Mathematical Practice of Implementing Mathematical Processes includes the skill "Explain how an approximated value relates to an actual value." Teachers should take early opportunities to discuss the relationship between an actual and approximated value. This could be done any time an approximated value from a graphing calculator is used. By emphasizing the relationship, teachers can build a foundation for the understanding of the relationship between Taylor Series and Taylor polynomials.