2019 BC5 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were asked to consider the family of functions given by $f(x) = \dfrac{1}{x^2 - 2x + k}$ where *k* is a constant.

Part a:

Students were asked to find the value of *k*, $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ is 6.

Part b:

Students were asked to find the value of $\int_0^1 f(x)\,dx$ for $k = -8$.

Part c:

For $k = 1$ students had to find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

Comments on student responses and scoring guidelines:

Part a: worth 3 points

Part (a) required work calculating $f'(x)$ in terms of x and k, setting $f'(0) = 6$, and then solving for *k*. The derivative calculation was awarded the first two of the three points in this part of the problem. The first of these two points was for a correct denominator of $f'(x)$. This could be shown during calculations and awarded the first point even in the presence of other errors. The second point was a report of a completely correct $f'(x)$. Something like $f'(x) = \dfrac{x^2 - 2x + k(0) - 1(2x-2)}{(x^2 - 2x + k)(x^2 - 2x + k)} = \dfrac{-2x + 2}{(x^2 - 2x + k)^2}$ was awarded the first point for the correct denominator, but despite the correct final form of $f'(x)$ was not awarded the second point due to the missing parentheses in the first numerator. Such work was eligible for the third point for the answer, $k = \sqrt{\tfrac{1}{3}}$. Other minor errors in calculating such as $f'(x) = \dfrac{2x-2}{(x^2 - 2x + k)^2}$ had a variety of effects on the scoring. In this example, the student earned the first point for the correct denominator, not the second point, and no consistent answer for the third point is possible since this would result in $k^2 = -\dfrac{2}{6}$. Even in the presence of a correct final answer, student work that linked with an equal sign any expressions not actually equal would not earn the third point.

Part b: worth 3 points

The evaluation of $\int_0^1 f(x)\,dx$ required partial fraction decomposition. It was very important to factor the denominator correctly. The incorrect factorization $x^2-2x-8=(x+4)(x-2)$ was read for a possible second point award for antiderivatives and no other points. All other incorrect factorizations resulted in 0 points in part (b). Incorrect constants as in $\frac{1}{x^2-2x-8}=\frac{1}{x-4}+\frac{1}{x+2}$ were read for a possible second point for antiderivatives and a possible third, consistent, answer point. If no absolute values were shown in the antiderivatives, the second point was not earned, but it was possible to earn the third point if "late" absolute values were shown. There were *many* examples of student work trying simplification and application of properties of logs that were not necessary to be considered and often cost students the third point.

Part c: worth 3 points

These three points required students to show they were working with an improper integral, calculating antiderivatives, evaluating using a one-sided limit, and declaring that the integral diverges. The first point was for an improper integral which could be indicated by something like the following:

$$\int_0^1 \frac{1}{(x-1)^2}\,dx+\int_1^2 \frac{1}{(x-1)^2}\,dx$$

OR

$f(x)$ has a vertical asymptote at $x=1$

OR

$\lim\limits_{x\to 1} f(x)=\infty$

OR

$f(x)$ has an infinite discontinuity at $x=1$

Emphasis here had to be on the *infinite* nature of the discontinuity. Thus the following responses were not sufficient to earn the first point:

$\lim\limits_{x\to 1} f(x)$ does not exist

OR

$f(x)$ has a discontinuity at $x=1$

OR

$f(x)$ is undefined at $x=1$

For the second point, the antiderivative(s) needed to be shown as in $\frac{-1}{x-1}$, $-(x-1)^{-1}$ or $-\frac{1}{u}$ if using the substitution $u=x-1$. The third point was awarded for a correct evaluation with work showing use of a one-sided limit and the correct conclusion of divergence. While only one of the two integrals had to be evaluated, if both were, both computations had to be correct. Any of the following work was acceptable for the third point, or the student could simply say that the limit does not exist (with the conclusion "diverges"):

$$\lim_{b\to 1^-}\left(-\frac{1}{b-1}-1\right)=\infty$$
$$\lim_{b\to 1^+}\left(-1+\frac{1}{b-1}\right)=\infty$$
$$\lim_{b\to 1^-}\left(-\frac{1}{b-1}\right)=\infty$$
$$\lim_{b\to 1^+}\left(\frac{1}{b-1}\right)=\infty$$

If limits were written in part (c) as "$=\frac{1}{0}$" or if $\infty-\infty$ was seen, students did not earn the third point.

Observations and recommendations for teachers:

(1) Using the quotient rule to calculate the derivative of a function $f(x)=\dfrac{c}{g(x)}$ is a good strategy since the numerator of the derivative contains the derivative of *c* which is 0, simplifying the calculation. Students should always use parentheses in the numerator when applying the quotient rule. A classic error in quotient rule calculation is reversing the terms in the numerator. Practice with a memorized form of the quotient rule is important. Students who rewrite the function expression as in $f(x)=c\left(g(x)\right)^{-1}$ seem to make more errors than if using the quotient rule calculation of $f'(x)$, perhaps because of the additional need for a chain rule computation with the accompanying need for parentheses.

(2) Students whose work culminates in something like $k=\sqrt{-\frac{1}{3}}$ or $c=\ln(-3)$ or $-2=e^{c}$ should see a red flag, but not panic. They should return to their first work, looking for simple things such as an arithmetic, parenthesis, or quotient rule reversal error. Too many students wrote $k=\sqrt{-\frac{1}{3}}$ and stopped work in part (a).

(3) It is not a bad idea to practice partial fraction decomposition as simply an algebraic operation. Presenting this to students simultaneously with calculation of an antiderivative (and maybe subsequent definite integral evaluation) may not be the best way for students to learn to be comfortable with just the algebra. The algebra works well when a fairly simple denominator of a rational expression can be nicely factored, but it does require practice.

(4) While it is certainly important for students to know and practice with properties of logs, using them to rewrite and simplify expressions, students should *not* apply these properties in order to simplify an answer on the AP Calculus Exam. An answer does not have to be simplified. Many students, wanting to report a simplified answer, lost a chance at a point because of minor errors.

(5) The antiderivative of $\frac{1}{f(x)}$, with $f(x)$ a linear function, should always be expressed as log of an absolute value. Absolute value notation needs to be present until it is clear that the argument of the log function is positive.

(6) Computing the value (or showing the divergence of) an improper integral as in $\int_a^b f(x)\,dx$ requires careful work using limits if $f(x)$ has an infinite discontinuity for any value of x on the interval $a \le x \le b$. These limits are one-sided unless, perhaps, *a* or *b* is $\pm\infty$. In part (c) the integrand was undefined at $x = 1$ requiring the integral to be broken into two parts. In one of these parts the antiderivative needed to be evaluated using limit notation and a one-sided limit as in $b \to 1^-$ and in the other $b \to 1^+$. This should be practiced in class. For example, $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)\Big|_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2}$ arrives at the correct value incorrectly. Since the integrand $\frac{1}{\sqrt{1-x^2}}$ has an infinite discontinuity at $x = 1$, a correct calculation verifying the result proceeds as follows:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1^-} \sin^{-1}(x)\Big|_0^b = \lim_{b \to 1^-} \sin^{-1}(b) - \sin^{-1}(0) = \frac{\pi}{2} - 0$$

There are many examples where ignoring the use of limit work still results in a correct final answer despite the lack of proper limit notation and calculation. Students should see examples such as this, as well as practice extensively showing work with one-sided limits.