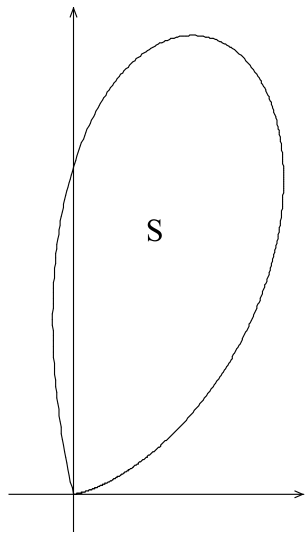
2019 BC-2 Dennis Wilson, Landmark Christian School

Problem Overview:



Students were given the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin(\theta^2)$ for $0 \le \theta \le \sqrt{\pi}$ and told that S is the area bound by the curve.

Part a:

Students were asked to find the area of S.

Part b:

Students were asked to determine the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin(\theta^2)$ for $0 \le \theta \le \sqrt{\pi}$.

Part c:

Students were told that a line with a positive slope m divides the region S into two equal regions with equal area. Students were asked to write an equation involving one or more integrals whose solution would give the value of m. Students were not required to solve the equation.

Part d:

Students were told that $A(k)$ was the area of the portion of the region S that was also inside the circle $r = k\cos\theta$. Students were asked to find $\lim_{k\to\infty} A(k)$.

<u>General scoring guidelines for the problems:</u>

<u>Part a:</u> (2 points)

The first point in this part and in part (b) is for an integral. The integral point requires two things, the correct bound and the correct integrand. Coefficients did not count as part of the integrand. The coefficient of ½ was not needed to earn the first point. Missing differentials were also not penalized in this problem.

The second point came for the answer. Students could not earn the second point without an integral. The student could earn the second point for the answer without earning the point for the integral. If the student had a presentation error from incorrect placement of parentheses, they would lose the first point but could earn the second with the correct answer.

If students gave an answer consistent with the calculator being in degree mode, then they lost this point. They were however inoculated from losing points in part (b) and (d) with consistent answers.

From the work in part (a), it was obvious to the reader whether the student could enter the question or not. Students who demonstrated knowledge of the area bound by polar curves would typically earn both points here and at least one, if not more, in other parts of the question. Some of these students failed to use the correct constant, using $\frac{\pi}{2}$ instead of $\frac{1}{2}$.

A significant number of students answered the question as if they were determining the area bound by the cartesian graph, failing to square the function $r(\theta)$ in the integrand.

<u>Part b:</u> (2 points)

The first point in this part was also for the integral. As in part (a), a correct coefficient was not necessary to earn this point. The second point for the answer required the presence of the integral. If the student made a presentation error due to misplaced parentheses in both parts (a) and (b), then the student would be inoculated in part (b) in the presence of correct answers.

Students who were familiar with the concept of distance for a polar curve had no difficulty earning these points. Some students however produced an immense amount of work involving the distance formula. While this method correctly produces the answer, many fell astray because of the large amount of algebra and parentheses required. Some students treated the problem as the length of a parametric curve and were not able to earn any points for part (b).

<u>Part c:</u> (3 points)

The first point in this part for equating polar areas required a structurally correct equation involving at least one polar area integral whose solution would yield a value for m.

The second point required students to have an integral whose limits included an angle which involved an inverse trigonometric function of m. This trigonometric relationship between the limit and the value of m did not need to be correct to earn this point. The point could be earned even if the integral did not calculate area. The first and second point were not linked.

The third point was earned for a correct equation. Students were only eligible for this point if they earned the first two points for this part. Presentation errors were deducted this point.

Many students were able to earn the first point only for part (c). These students used m as limits for their integrals without involving any trigonometric function. Students who understood that the limits of the integral should be an angle often created a second equation for m, relating it to a second variable for the appropriate angle in the limits of their integral. This system of two equations earned all three points.

Part d: **(2 points)**

The first point was awarded in this part if the student had a definite integral with the limits of 0 and $\frac{\pi}{2}$. A three decimal place approximation for $\frac{\pi}{2}$ could be used here to earn the first point. If this approximation was used in the calculation of the integral though, it produced an incorrect value for the area. Students could also earn this point by subtracting an integral with limits from $\frac{\pi}{2}$ and $\sqrt{\pi}$ from their answer given in part (a).

The second point could only be earned if the first point was earned. If the student earned the first point and gave the correct value of the area to three decimal places, then the student earned this second point.

Very few students earned points in part (d). Almost without exception, the students who earned full credit had drawn a sequence of circles on the paper for increasing values of k. Some students unfortunately understood the consequence of the limit but misread the problem. They found the area inside the region S but outside the circle $r = k\cos\theta$.

Observations and recommendations for teachers:

(1) The bulk of the points for this question were connected to the learning objective for calculating areas of regions defined by polar curves using definite integrals (CHA5-D). The first two points for this question could be earned simply from knowledge of the formula for making such a calculation. This is the third consecutive year for the BC exam to include a Free Response Question involving a polar curve or curves. Teachers should take time before the exam to review what can seem to be an isolated concept in the curriculum. Teachers should also consider ways to weave the concepts of polar curves throughout their curriculum instead of making slope and area for these curves a separate and distinct idea. The fact that the concept is only on the BC exam should not relegate these concepts to stand isolated at the end of their course.

(2) The concept of distance between two polar curves or now, the distance between the curve and the origin has been asked in three of the last six years. (2014, 2017, and 2019) While textbooks do not treat these types of problems in abundance, teachers should consider making the solution

of such problems a key aspect in their instruction of polar concepts. Students should understand the value of the variable r is defined as a distance from the origin.

(3) Dividing a region bound by a polar curve into equal areas is also a common question. Even though new variations are being added (e.g. a line of slope m), most students do well setting up an equation which will earn points. Some students however still split the entire region into two definite integrals whose sum is the total and yet are not equal to each other. This mistake should be avoided when the properties of a definite integral are first introduced. The property

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

should be explored graphically with various values of c, so students will not fall victim thinking the integral is being split into equal parts.

(4) Students are taught well before entering the calculus classroom the slope of a line gives the tangent of the angle made by the line and the x-axis. Calculus teachers should use reinforce this student knowledge not only in the study of polar curves but also in their study of tangent lines. The mathematical practice “Connecting Representations” from the new Course and Exam Description (CED) incudes the skill “Identify common underlying structures in problems involving different contextual situations.” As each point in the Free Response Questions is now keyed to one of these skills, teachers should expect more questions to incorporate various concepts across the curriculum. Classroom instruction should be analogously adapted.

(5) While the exam requires students only to perform four operations with the handheld graphing calculator, the course itself requires more robust use of technology. The Mathematical Practice of Justification includes the skill “Apply technology to develop claims and conjectures.” While the CED states that this skill is not assessed on the exam, many students used this skill to develop their understanding and solution to part (d). The student should become proficient with the many features of the handheld graphing calculator in order to explore mathematical ideas quickly on a Free Response Question.

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