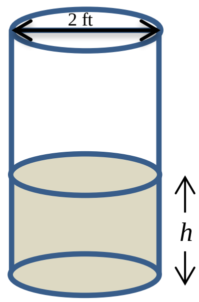
2019 AB/BC4 Marshall Ransom, Georgia Southern University

Problem Overview:



The cylindrical barrel shown above with diameter 2 feet contains collected rainwater. Water is draining out through a valve (not shown) in the bottom of the barrel. The rate of change of the height *h* of the water in the barrel with respect to time *t* is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ where *h* is measured in feet and *t* is measured in seconds. Students were given that the volume of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.

Part a:

Students were asked to find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet and indicate units of measure.

Part b:

Students were asked if the rate of change of the height of the water is increasing or decreasing when the height of the water is 3 feet and to explain their reasoning.

Part c:

At time $t = 0$ seconds the height of the water is given as 5 feet. Students were asked to use separation of variables to find an expression for *h* in terms of *t*.

Comments on student responses and scoring guidelines:

Part a: worth 2 points

The first point was for calculating $\frac{dV}{dt}$ in terms of *h*. This is perhaps most easily done by substituting 1 for *r* yielding $V = \pi h \rightarrow \frac{dV}{dt} = \pi \frac{dh}{dt}$. This could also be expressed as $\frac{dV}{dt} = \pi\left(-\frac{1}{10}\sqrt{h}\right)$. Students could also report a correct product rule as in $\frac{dV}{dt} = 2\pi r r' + \pi r^2 h'$ and earn this first point. The most common error using the product rule was omitting r'. Another common error was assuming that $r = 2$. Students *showing* $r = 2$ somehow in their work arrived at $V = 4\pi h \rightarrow \frac{dV}{dt} = 4\pi \frac{dh}{dt}$ and could earn the first point but not the second. The second point was for the answer with units and was only awarded for the correct $-\frac{\pi}{5}\ {ft^3}/{s}$. Notation could play a role as in $\frac{dV}{dt} = \pi \frac{dh}{dt} \rightarrow \frac{dV}{dt} = \pi - \frac{1}{5}\ {ft^3}/{s}$ where it is not clear if the student is subtracting from π or multiplying because of missing parentheses. This last example would not earn the second point.

Part b: worth 3 points

The rate at which the height is changing is the first derivative of *h*. In order to determine whether or not this is increasing or decreasing, $\frac{d^2h}{dt^2}$ has to be calculated. The first step in this calculation is to compute $\frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}}$. This earned the first point. Work showing the chain rule being applied earned the second point. Many students had $-\frac{1}{20\sqrt{h}}$ somewhere in their work, earning the first point, but neglected to complete the calculation by applying the chain rule as in $\frac{d^2h}{dt^2} = \frac{-1}{20\sqrt{h}}\frac{dh}{dt}$ and could not earn the second or third points. The third point was for the answer with reasoning. The mere presentation of a positive second derivative of *h* accompanied by the word "increasing" could earn this point. Unfortunately, many students went further to explain in words beyond the word "increasing" and often did not earn this third point. An arithmetic error in calculating the value of $\frac{d^2h}{dt^2}$ kept students from earning the third point. There is no required arithmetic or simplification here, only the search for and display of a positive value of $\frac{d^2h}{dt^2}$. For example, "$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{-1\sqrt{h}}{10} \rightarrow$ increasing" earned all three points in part b.

Part c: worth 4 points

Eligibility for points in solving a separable differential equation on the AP Calculus Exam always requires a correct or "almost" correct separation of variables. For example, $\frac{dh}{\sqrt{h}} = -\frac{1}{10}dt$ and $\int \frac{1}{\sqrt{h}} = \int -\frac{1}{10}$ earned the first point for separation. Work showing $h^{-1/2}dh = kdt$ where $k = \pm\frac{1}{10}, \pm 1, \pm 10$ was still eligible for the second and third points even using one of the five constants that are not correct; In fact, $k = -\frac{1}{10}$. Students launching into antiderivatives as in $2\sqrt{h} = -\frac{1}{10}t$ or $2\sqrt{h} = -\frac{1}{10}t + C$ earned both the first point for separation and the second point for antiderivatives. Both antiderivatives needed to be correct in order to earn the second point. If an antiderivative showed h^1 or $\ln(h)$, students could not earn any of the second, third or fourth points. Not showing one of these forms of a bad antiderivative, students having at least one of the antiderivatives correct could earn the third point for having $+C$ in a timely manner *and* trying to use the initial condition correctly. Students showing $\frac{dh}{\sqrt{h}} = -\frac{1}{10} \rightarrow \sqrt{h} = -\frac{1}{10}t + C \rightarrow C = \sqrt{5} \rightarrow h = anything!$ earned the first point, not the second, because of the incorrect antiderivative on the left side, the third point because $C = \sqrt{5}$ comes from correctly using the initial condition, but were not eligible for a correct answer fourth point. Most students more obviously showed the use of $t = 0$ and $h = 5$ if they made it this far in the work, but solving for our "$+C$" was not part of the third point, it being considered part of the fourth point for the expression for *h*. The reason that $C = \sqrt{5}$ is looked at by readers in this last example is because it provides evidence of *using* the initial condition. A trivialization of the problem kept students out of all four points. This could be shown using the separation of $\sqrt{h}dh = -\frac{1}{10}dt \rightarrow anything$ where not having a negative exponent on *h* was considered too much trivialization even if correct work followed.

Observations and recommendations for teachers:

(1) The rate of change of a function is the first derivative of that function with respect to time. In part (a) there were two variables, which would require the calculation to use a product rule had not the given information indicated that *r* is constant. In that case, the value of *r* may be substituted before the derivative calculation. Giving students the value of the *diameter* created difficulties for a number of students in this part of the problem. Many students used the value of the diameter as the value of the radius. Students should read the problem carefully.

(2) The reporting of a numerical answer on the AP Calculus Exam does not require any arithmetic simplification. Students who automatically yield to the urge to simplify often lose a point on the exam and could do so in all three parts of this question. Note that $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{-1\sqrt{h}}{10}$ indicates that the value of h in $\left.\frac{d^2h}{dt^2}\right|_{h=?}$ is completely irrelevant and that $\left.\frac{d^2h}{dt^2}\right|_{h=?} > 0$ in the part (b) calculation. No simplification is required.

(3) In part (b), students were asked to determine if a rate (a first derivative) was increasing or decreasing. All that needs to be shown to justify the answer is a correct calculation, clearly either positive or negative. The sign of the result of that calculation is assumed to be justification of the choice of either "increasing" or "decreasing." Writing more explanation invites incorrect information such as adding a criterion for justification that is not required. Students writing additional explanatory language after displaying calculations leading to a positive second derivative often lost the third point in part (b).

(4) When solving a separable differential equation and showing work, students should be taught to include the differentials. There must be only one "variable" on each side of the equation resulting from separation. After this separation, this is merely an exercise in antidifferentiation and solving for C using the given initial condition. On the AP Calculus Exam, the appropriate (timely) appearance of C and the substitution of the given initial condition has for several years before 2019 been awarded one point after readers examine the separation and the antiderivatives. Subsequent work in solving for C and finding an expression for the function can often involve several steps, and the result is worth only one point. No algebra or simplification need be done until after C has been determined. It is not a bad test taking strategy to save this last work (for only one point) until the entire exam has been worked because of the time and work it can take.

(5) An all too common error is for students to report that the antiderivative of any "1 over A" function is the natural log of A. This common error should be shown and shunned more than once when teaching after students have knowledge of the fact that $\int \frac{dt}{t} = \ln|t| + C$. In part (c), this error showed its ugly head when students wrote that $\int \frac{dh}{\sqrt{h}} = \ln\left|\sqrt{h}\right| + C$. This error took students out of all points save the possible award of the one point for a correct separation.

(6) The final answer giving an expression for h in part (c) did not require that much attention be paid to the domain of h. Because of the context of the problem, both $h, t \geq 0$ eliminating any concerns that might have arisen because of the presence of $\sqrt{h}$. It is often the case that the domain of the final expression must be considered and has at times appeared as a specific question on the AP Calculus Exam. Functions such as $\ln|x|$, $\sin^{-1}(x)$ or $|\text{expression}|$ require a closer look. Also, a final algebraic step as in solving for y if we have $y^2 = \text{expression}$ can require a look at which branch, $\pm\sqrt{\text{expression}}$, should be chosen. Problems involving these considerations should be used by teachers in the classroom. For some examples from past AP Calculus Exams see the following: 2006AB5 part (b), (b) 2010AB6 part (c), 2011AB/BC5 part (c), 2013AB6 part (c), and 2014AB6 part (c).