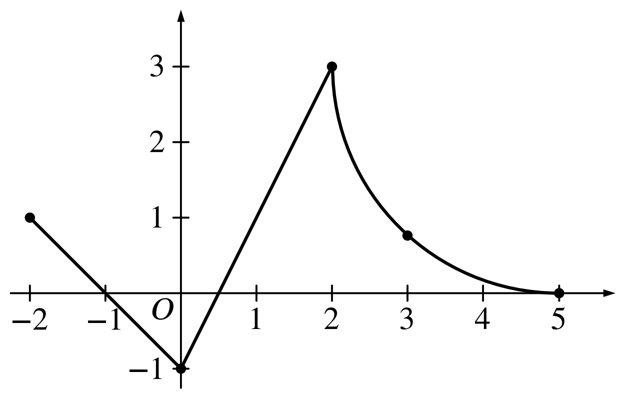
2019 AB/BC-3 Chuck Garner, Rockdale Magnet School

Problem Overview

The student is given a portion of the graph of the continuous function f (shown below) which consists of two line segments and a quartercircle. The function f is defined on the interval $-6 \le x \le 5$. The student is told that the quartercircle is centered at $(5, 3)$ and the point $(3, 3 - \sqrt{5})$ lies on the graph.



The graph of f

Part a

Students were told that $\int_{-6}^{5} f(x)\, dx = 7$ and were asked to show the work that leads them to the value of $\int_{-6}^{-2} f(x)\, dx$.

Part b

Students were asked to evaluate $\int_{3}^{5} (2f'(x) + 4)\, dx$.

Part c

Students were given a new function g defined by $g(x) = \int_{-2}^{x} f(t)\, dt$ and were asked to find the absolute maximum of g on the interval $[-2, 5]$.

Part d

Students were asked to evaluate $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Comments on Student Responses and Scoring Guidelines

Part a

The problem was about area properties of definite integrals and three points were available for the student. Students had to show that they could handle these properties by writing some equivalent form of the statement

$$\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx.$$

Showing this property in some form earned the first point. To award the first point, the Readers were given some leeway with notation: missing dxs were ignored since the presence of an equals sign or the addition or subtraction of a second integral was assumed to close the first integral.

Then students had to use the fact that $\int_{-6}^{5} f(x)\,dx = 7$ along with the computation of the areas of triangles and a square minus a quartercircle to compute $\int_{-2}^{5} f(x)\,dx = 11 - \frac{9\pi}{4}$. Correctly computing the areas earned the second point. The area values did not need to be simplified; many correct area values were left unsimplified and earned the point. However, a student simplifying a correct expression to an incorrect final answer did not earn the point. Finally, they had to show that the definite integral requested was equal to $-4 + \frac{9\pi}{4}$, and this earned them the third point.

If the area under the graph was computed incorrectly, the student did not earn the second point, but was still eligible to earn the third point, with one exception: The student had to show that the quartercircle was handled by taking the area of the square and subtracting the area of the quartercircle. This should be expressed as $9 - \frac{9\pi}{4}$. Any lack of consideration for this (i.e., writing $\frac{9\pi}{4}$ by itself, or writing $\frac{9\pi}{4} - 9$, or π missing completely from the computation) did not earn the second or third point. If a student used 3.14 for π, the student did not earn the second point but could still earn the third (although this did not happen as students did not subsequently report an answer correct to three decimal places).

A minimal response which earned all three points is the following.

$$7 - \left(11 - \frac{9\pi}{4}\right)$$

This demonstrates a sufficient knowledge of the properties of definite integrals; the area in the graph is computed correctly; and the final correct answer is reported.

Part b

This problem could earn the student two points. The first point is earned by demonstrating appropriate use of the Fundamental Theorem of Calculus. Writing an antiderivative of $f'(x)$ as $f(x)$ and then evaluating it as $f(5) - f(3)$ was all that had to be done to earn this first point. The second point was earned for doing everything else about this problem correctly: handling the integration of 4 from $x = 3$ to $x = 5$, the coefficient of 2, computing the values of $f(5)$ and $f(3)$, and any simplification of the final answer.

A minimal response which earned both points was the following.

$$20 - \left(6 - 2\sqrt{5} + 12\right)$$

This shows that the appropriate antiderivatives were found and evaluted.

Part c

Three points are available to the student in part (c). To earn the first point, the student had to show knowledge once again of the Fundamental Theorem of Calculus by writing $g'(x) = f(x)$, by writing $g' = f$, or by expressing this relationship in words. To earn the second point, the student must indicate, among a list of possible x-coordinates for extrema, that one of those x-coordinates is $x = -1$. Any indication of the consideration of $x = -1$ earned the point, such as writing $x = -1$ anywhere in their work for part (c), including -1 in a list of critical points, writing $g(-1)$, or writing $\int_{-2}^{-1} f(t)\,dt$.

To earn the third point, the student had to declare the correct maximum value, which is the area under the graph, $11 - \frac{9\pi}{4}$. This maximum value had to be clearly indicated by the student; saying "the maximum occurs at $x = 5$" or reporting the ordered pair $(5, 11 - \frac{9\pi}{4})$ did not earn the third point. Importing incorrect areas from part (a) were accepted as long as the incorrect area was greater than $g(-1) = \frac{1}{2}$. However, a student who did not explicitly rule out $x = \frac{1}{2}$ as a possible maximum did not earn the third point.

Part d

To earn the only point available, the student just had to plug in $x = 1$ and evaluate $f'(1)$. Since $f'(1)$ can be obtained from the graph as being equal to 2, a correct response with no simplification such as

$$\frac{10^1 - 3 \cdot 2}{1 - \arctan 1}$$

earned the point. There was no need to evaluate $\arctan 1$. A student who was "over-zealous" with limit notation and wrote

$$\lim_{x \to 1} \frac{10^1 - 3 \cdot 2}{1 - \arctan 1}$$

did not earn the point, as this student still has not reported the value of the limit. If $\arctan 1$ was evaluated, it must be $\frac{\pi}{4}$, not $45°$, as the inverse trigonometric functions are defined only with ranges in radians. A student using degrees (or making simplification errors) lost the only point available.

Observations and Recommendations for Teachers

(1) Students should know how to find areas of simple geometric figures. Many students could not recall the area of a circle. Some students could not find the area of a triangle. Give students plenty of practice with graphical problems where they need to calculate the areas of triangles, trapezoids, semicircles, and quartercircles. Students should also understand that area is calculated between the curve and the x-axis. Many students calculated areas between the curve and the line $y = -1$. Some calculated areas above the

curve. Others attempted to count the grid squares and estimate the area. (Any such estimations did not earn points in part (a).)

(2) Students should read the problem. In part (b), students who computed the definite integral had to compute $f(3)$. The value of f at $x = 3$ was given in the problem as $3 - \sqrt{5}$, and yet students did not use it. Without including this value in the computation of the answer, students did not earn the answer point. Again, some students tried to estimate the value of $f(3)$ from the graph as 0.75 or 0.8, but these estimations also did not earn the point.

(3) Students should be discouraged from using the First Derivative Test when confronted with a function on a closed interval. In part (c), students had to work very hard to justify the correct maximum when using the First Derivative Test, since the correct maximum occurred at an endpoint. The best approach is to list all possible critical points, and then compute the function values at those points, and then declare which one of those function values is the largest. Students should be practicing extrema problems on closed intervals in class. Problems of this type have appeared on recent AP exams, and students should be able to handle them.

(4) Students should read the problem. A remarkably large number of students, even though asked to find the maximum in part (c), instead found the minimum, and declared their answer as the minimum.

(5) Students should avoid arithmetic. Many students had great work for part (a) which did not earn the answer point due to sign errors. Unsimplified work is scored by Readers and helps the student avoid such errors.

(6) Students should know simple trigonometric values. In part (d), even though an unsimplified $\arctan 1$ was shown, many students carried forth and evaluated it, and many of those who evaluted it did so incorrectly as 0, $\frac{\pi}{2}$, $\frac{\sqrt{2}}{2}$, or "undefined". Some indicated that they had never seen "arctan" before. Still other students believed that $\arctan 1$ could be equal to either $\frac{\pi}{4}$ or $\frac{5\pi}{4}$, which is incorrect. As this problem was only worth one point, any error means that the point was not earned.