

Problem Overview:

2
3
4
5
6 Functions f , g and h are twice differentiable functions with $f(2) = g(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is
7 tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.
8

Part a:

9
10 Students were asked to find $h'(2)$.
11
12

Part b:

13
14
15
16 The function a is given by $a(x) = 3x^2h(x)$. Students were asked to write an expression for $a'(x)$ and also to
17 find $a'(2)$.
18
19

Part c:

20
21
22 It is stated that the function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \rightarrow 2} h(x)$ can be
23 evaluated using L'Hopital's rule. Students were asked to use $\lim_{x \rightarrow 2} h(x)$ to find $f(2)$ and $f'(2)$.
24
25

Part d:

26
27
28 It is known that $g(x) \leq h(x)$ for $1 < x < 3$ and that k is a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$.
29 Students were asked if k is continuous at $x = 2$ and to justify their answers.
30
31
32
33

Comments on student responses and scoring guidelines:**Part a:**

34
35
36
37
38
39 This part was worth one point only and only for the correct answer. There is no work to show and the
40 answer did not have to be attached to mathematical notation as in $h'(2) = \frac{2}{3}$. Students reporting incorrect
41 answers were eligible to use that in other parts of the problem. The most commonly reported incorrect
42 answer was 4. Many students did show work which could usually be ignored by readers. Very rarely,
43 student work was shown to calculate the correct answer in a manner that was incorrect resulting in no point
44 for part (a).
45
46

47 **Part b:** worth 3 points

48

49 Calculating the derivative of this function requires the product rule. The first point was for showing the form
50 of a product rule calculation and required four things: a sum of two terms, h in one term, h' in the other,
51 and multiplication of each term by a polynomial (which could be a constant). If this was shown, students
52 earned the first point. If students did not earn this first point because of committing common product rule
53 errors given by $a'(x) = 6xh'(x)$ or $a'(x) = 3x^2h'(x)$ or any other error that did not meet the four criteria, they
54 were not going to earn the second and third points. Special case: a number of students substituted
55 $y = 4 + \frac{2}{3}(x-2)$ for h before calculating the derivative. Such students were not awarded the first and
56 second points, but were eligible for the third point for a correct value of $a'(2)$. Another example:
57 $a'(x) = 6x^2h(x) + 3x^3h'(x) \rightarrow a'(2) = 112$ would earn the first point for correct product rule form and the
58 third point for an $a'(2)$ answer consistent with that form, but would not earn the second point for a correct
59 expression of $a'(x)$. A few students expressed the derivative numerically perhaps as $36 \cdot 4 + 24 \cdot \frac{2}{3}$ and could
60 only earn the third point. It is most wise to express a derivative using function notation and the variable on a
61 test requiring work to be shown.

62

63

64 **Part c:** worth 4 points

65

66 As might be expected, students launched into the work at different places and readers needed to search for
67 work that earned points. The first point was for finding a connection between $\lim_{x \rightarrow 2} h(x)$ and the number 4.

68 Some intervening work may have contained errors, but this connection ultimately shown earned the first
69 point. Since L'Hopital's rule can be used to calculate $\lim_{x \rightarrow 2} h(x)$ and $\lim_{x \rightarrow 2} (x^2 - 4) = 0$, students needed to

70 show that they were working with $1 - (f(x))^3 = 0$ in order to arrive at $f(2) = 1$. This earned the second

71 point. (See **Observations and recommendations for teachers #4** below). The third point was for

72 applying L'Hopital's rule. Student work needed to show three things: limit notation, a quotient, and
73 convincing evidence of the attempt to calculate derivatives of both the numerator and denominator. Forms

74 such as the correct $\lim_{x \rightarrow 2} \frac{2x}{-3(f'(x))^2 f(x)}$ or the incorrect $\lim_{x \rightarrow 2} \frac{2x-4}{1-3(f'(x))f(x)}$ earned this point. Using $= \frac{0}{0}$

75 attached to any work would not earn the third point, but such students were still eligible for the fourth point.

76 Only the correct $-\frac{1}{3}$ was accepted for the award of the fourth point. The eligibility for this fourth point was
77 earning the third point. The only exceptions to that eligibility for the fourth point were if the third point were

78 denied solely because of the lack of limit notation or the connection $= \frac{0}{0}$.

79

80

81 **Part d:**

82

83 To earn just one point, students had to show/reference five things: "yes" for the answer; both h and g are
84 continuous; $\lim_{x \rightarrow 2} h(x) = \lim_{x \rightarrow 2} g(x) = 4$; $\lim_{x \rightarrow 2} k(x) = 4$; and $k(2) = 4$ because $h(2)$ and $g(2)$ both equal 4. If the

85 Sandwich Theorem was named, some colloquialisms such as "Pinch Theorem" or "Squeeze Theorem" were
86 acceptable, but no name was required. Specifically naming an incorrect theorem such as Rolle's or IVT

87 disqualified the student from this point. Readers did not have to read much in part d. While there was lots of

88 writing, a quick look could often easily discern that one of the five criteria was missing. I personally read
89 hundreds of student responses to this question and only saw one that earned the point in part d.

90
91
92
93
94 **Observations and recommendations for teachers:**

95
96 (1) The basic fact being applied in part (a) is that a line tangent to the graph of a function h at a point where
97 $x = a$ has a slope at that point equal to the derivative of h at that point. But the expression for the tangent
98 line function is not equivalent to the function. A number of students substituted the expression for the
99 tangent line function for h before computing the derivative. This will, of course, have the correct derivative
100 of h at that point, but not necessarily anywhere else. More importantly, the tangent line is not h , meaning
101 that the student after substituting is not working with the given function.

102
103
104 (2) The product and quotient rules are very basic computational skills. Far too many students reported a
105 single term when calculating $a'(x)$ in part (b), indicating a lack of basic computational knowledge. The
106 exam does require such computational facility, as well as for basic antiderivatives and using u -substitution.

107
108
109 (3) Students should read the question carefully, both before and after work. In part (b), both $a'(x)$ and
110 $a'(2)$ were requested. While strong students can calculate quickly toward a value of $a'(2)$, leaving out work
111 on the paper showing $a'(x)$ means that they were not answering part of the given question. Messages:
112 substitute values for x after calculating an expression for a derivative (or antiderivative); and when done
113 with work, read the question again to be certain that all that is requested has been addressed.

114
115
116 (4) Technically it is the $\lim_{x \rightarrow 2} 1 - (f(x))^3$ that equals 0. But in order to calculate $f(2)$ an equation can be
117 used. Some students did show $1 - (f(x))^3 = 0$ and concluded that $f(x) = 1$. The value of 0 is only
118 guaranteed to be correct at the point where $x = 2$. Such students did not earn this point in part c.

119
120
121 (5) L'Hopital's rule on the exam has for the last few years has only involved the form $\frac{0}{0}$. But it is *never*
122 correct to report that a limit $= \frac{0}{0}$. A limit is a value and $\frac{0}{0}$ is an indeterminate *form*, not a value. This will
123 always cost the student an opportunity to earn at least one point somewhere when trying to apply L'Hopital's
124 rule. Also potentially penalizing the student is poor notation linked with an equal sign. For example, after
125 differentiating numerator and denominator, writing $\lim_{x \rightarrow 2} h(x) = \frac{2x}{-3(f(x))^2 f'(x)}$ is not correct because limit
126 notation needs to be attached to the expression $\frac{2x}{-3(f(x))^2 f'(x)}$. Limit notation does not "go away" until
127 values are being calculated.

128
129

130 (6) The problem in part (d) is very difficult simply because there are so many things to show. This would
131 make for a good class discussion. First, notice that this is a “Yes or No” question that needs to be answered.
132 Second, working with the sandwich theorem requires limits of the functions on the left and right sides of the
133 “sandwich”. Thus limit notation must be shown. Third, this is about continuity which exists when the limit
134 is the same as the value of the function. The value of the function must be verified as well as the limit and
135 then these must be connected. Fourth, values for any function that come from a given limit value only exist
136 if that function is continuous at that point. “Justify your answer” is a request for including more information
137 such as the fact that a function is continuous (usually because it is differentiable) in order to state that the
138 value of the function at a point is the same as the value of the limit at that point.