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Problem Overview

The student is given a table indicating certain values of the velocity of particle P, which is moving along the x-axis. The student is told that the differentiable function v_P models the velocity of P, and that P is at the origin at $t = 0$.

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	−29	55	48

Part a

Students were asked to justify why the acceleration $v'_P(t)$ must be zero at least once on the interval $[0.3, 2.8]$.

Part b

Students were asked to approximate $\int_0^{2.8} v_P(t)\, dt$ using a trapezoid sum on the intervals $[0, 0.3]$, $[0.3, 1.7]$, and $[1.7, 2.8]$.

Part c

Students were told about another particle, Q, which also moves along the x-axis, and that its velocity is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour on the interval $0 \le t \le 4$. Students were asked to find the interval when the velocity of Q is at least 60 meters per hour. Then the students were asked to find the distance traveled by Q during that interval.

Part d

Students were given that at time $t = 0$, Q had position $x = -90$. Then, using part (b) and the function $v_Q(t)$ from part (c), students were asked to approximate the distance between P and Q at time $t = 2.8$.

Comments on Student Responses and Scoring Guidelines

Part a

For part (a), the student could earn two points. There were two approaches the student could use.

The Mean Value Theorem Approach. There was one point earned for stating that $v_P(2.8) - v_P(0.3) = 55 - 55 = 0$, whether that was in a difference quotient or not, or the student could express this in words. However, to earn the point, the students had to write that $v_P(2.8) - v_P(0.3) = 0$ or $55 - 55 = 0$; in other words, the student had to indicate that they knew that the evaluation was zero. To earn the second point, the student had to declare that $v_P(t)$ is continuous and conclude that $v'_p(t) = 0$ for some t in the interval $[0.3, 2.8]$. They did not need to cite the Mean Value Theorem, but it was helpful for the Readers. (Rolle's Theorem was also perfectly acceptable, and therefore the student indicating that $v_P(0.3) = v_P(2.8)$ instead of showing the difference is 0 also earned the first point.)

A minimal response which earned both points was

$$\text{Since } v_P(0.3) = 55 = v_P(2.8) \text{ and } v_P(t) \text{ is continuous, } v'_P(t) = 0.$$

The Extreme Value Theorem Approach. There was one point earned for stating that $v_P(0.3) > v_P(1.7)$ **and** $v_P(1.7) < v_P(2.8)$. Saying that "the velocity went from 55 to -29 and back" was not sufficient. However, saying that the velocity changed signs from positive to negative to positive was sufficient, as it is the sign change that indicates the presence of an extreme, not the actual function values. Students had to make this connection clear. The second point was earned for stating that $v_p(t)$ is continuous; that $v_P(t)$ has a minimum on the interval $[0.3, 2.8]$; and that $v'_p(t) = 0$ on the interval.

Every approach in which the student tried to use the Intermediate Value Theorem was rejected. See the "Observations and Recommendations for Teachers" below for information about rejected approaches.

Part b

The student could earn one point for a correct trapezoid sum. We wanted to see at least three products, at least three terms, and a sum. Work showing only $8.25 + 18.2 + 14.3 = 40.75$ was not sufficient to earn the point, even though this is correct, since the products are not shown. Readers had to see the student pulling the values from the table; that is, leaving an expression with $v_p(1.7)$ unevaluated was not accepted. The student could leave the sum unsimplified, but if evaluated, it must be correct. Indeed, since there is only one point available, everything the students showed must be correct. An average of a left-hand and a right-hand Riemann sum was accepted as well.

Part c

There were three points available for the student to earn in part (c). To earn the first point, the student had to write the interval over which $v_Q(t) \geq 60$. This could be expressed as $[1.866, 3.519]$, $(1.866, 3.519)$, or in words as "from time $t = 1.866$ to time $t = 3.519$". Not accepted was "$t = 1.866, 3.519$" since this does not indicate an interval. Readers were not allowed to use at the limits of integration in the definite integral of $v_Q(t)$ to see if students earned this point. The problem specifically asked for the interval, and the student needed to declare it apart from the definite integral.

The student earned the second point by setting up a definite integral for the distance particle Q traveled. Readers were given some leeway here. Since the problem stated an interval of $0 \leq t \leq 4$, Readers gave the point to any definite integral $\int_a^b v_Q(t)\,dt$ where $0 \le a < b \le 4$. Since the function $v_Q(t)$ is always positive, the student could integrate $v_Q(t)$ or $|v_Q(t)|$.

To earn the third point, the student had to report the correct value of the distance Q traveled, 106.108 or 106.109. No other value earned the point, and the value must be attached to a definite integral.

Part d

Three points could have been earned for this problem. The first point was earned if the student simply wrote $\int_0^{2.8} v_Q(t)\,dt$ somewhere on the paper. This could have been included in another expression, but then missing dts could cause difficulty. Both

$$-90 + \int_0^{2.8} v_Q(t) \qquad \text{and} \qquad \int_0^{2.8} v_Q(t)\,dt - 90$$

earned the point, but

$$\int_0^{2.8} v_Q(t)\ -90$$

did not since Readers make the assumption that the differential appears at the end of the expression. Thus the expression $\int_0^{2.8} v_Q(t)\ -90$ is taken to mean $\int_0^{2.8}(v_Q(t)\ -90)\,dt$ which is incorrect.

The second point was earned if the student declared 45.937 or 45.938 as the position of Q. No other answer was acceptable; indeed, the only other form of answer which earned this point was $-90 + 135.937$ or $-90 + 135.938$.

The third point was earned if the student showed the difference in the positions of P and Q. Any answer the student had for the position of P from part (b) was accepted in this computation, as was any position just computed for the position of Q. The student only earned this point if the values used in the subtraction of $x_P(t) - x_Q(t)$ were declared on the paper. However, the student could not earn this point if the definite integral $\int_0^{2.8} v_Q(t)\,dt$ was not on the paper to justify the calculation of the position of Q. The student was also asked to specifically provide the positive difference, so any negative answers did not earn the point. Some students tried to use the distance formula to compute this. Nearly all of these students expressed the positions of P and Q as the ordered pairs $(2.8, 40.75)$ and $(2.8, 45.938)$, and then computed

$$\sqrt{(2.8 - 2.8)^2 + (40.75 - 45.938)^2}.$$

This was considered correct, since this expression evaluates to the correct difference, and the values in the subtraction are declared.

Observations and Recommendations for Teachers

(1) Students should be trained to deal with Mean Value and Rolle's without being given a closed-form expression of a function. This problem was a perfect set-up for use of the MVT or Rolle: two equal function values as the endpoints of the interval over which you are asked to prove that the derivative of the

function is zero. Many students did not earn these two points because many students tried to use the IVT. But this led to two types of mistakes. The first type is that students forgot they were to show that $v'_P(t)$ is zero. Instead they proved that $v_P(t)$ is zero using IVT. The second type was more subtle, and it went something like this.

> On the interval $[0.3, 1.7]$, the function v_P decreases, so $v'_P < 0$. On the interval $[1.7, 2.8]$ the function v_P increases, so $v'_P > 0$. Then by the IVT, there is a point on $[0.3, 2.8]$ where $v'_P = 0$.

The problem with this argument is two-fold. First, saying that v_P is decreasing on $[0.3, 1.7]$ is taken to mean that the function is decreasing on the entire interval, and unfortunately, we do not know that. Second, in order to use IVT on $v'_P(t)$, we must know that $v'_P(t)$ is continuous, and we do not know that either. Just because $v_P(t)$ is differentiable does not imply that the derivative of $v_P(t)$ is continuous — the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is such a differentiable function with a discontinuous derivative. For these reasons, IVT arguments did not earn both points (unless the student happened to have noted $v_P(2.8) - v_P(0.3) = 0$ or $v_P(0.3) > v_P(1.7)$ **and** $v_P(1.7) < v_P(2.8)$, in which case they still earned the first point).

(2) Students should use reasoning based on theorems and definitions. Many students in part (a) and part (b) based reasoning on graphs that the student provided. In many cases students claimed that "the graph must look like this, so there has to be a minimum." Any graph the student provides is generally ignored by Readers when looking for justification. In part (b), many students attempted to draw trapezoids on top of a graph they drew themselves, which were generally incorrect.

(3) Students should use the correct theorem name when citing a theorem. There were instances where a student correctly used the Mean Value Theorem approach to the problem, but then said they were using the Intermediate Value Theorem. There also instances where a student used the Extreme Value Theorem approach, but then claimed the results followed by the Mean Value Theorem. Readers took the declaration of the theorem name as the approach the student intended to use. A student declaring the incorrect theorem could only earn the first point in part (a).

(4) Students should know how to find the area of a trapezoid. Probably 60% of students missed this point because they did not know how to compute the area of a trapezoid or because they performed incorrect arithmetic (on a calculator problem). Problems involving trapezoid sums have been asked on recent AP Exams, and will continue to be asked, and all students should be familiar with them.

(5) Students should adhere to the three-decimal-place rule. Perfectly fine work was ruined by decimal presentation errors. Even though a student will only ever lose a maximum of one point for a decimal presentation error (no matter how many times it happens), rounding incorrectly is a different error. In part (c), consider this first solution, seem on many student papers.

The interval is $[1.866, 3.52]$ and $\int_{1.866}^{3.52} v_Q(t)\,dt = 106.169$.

There is a decimal presentation error in the right-hand endpoint of the interval; it should be 3.519. Unfortunately, the student then used 3.52 in computing the distance that Q travels and got an incorrect answer. The student only earns 1 point for writing the definite integral.

Now consider another solution, also seen on many papers.

The interval is $[1.867, 3.519]$ and $\int_{1.867}^{3.519} v_Q(t)\,dt = 106.11$.

There are no decimal presentation errors in the interval. That is because 1.867 for the lower bound of the interval is simply incorrect. The value is 1.866181, so the student who wrote 1.867 wrote an incorrect interval. However, it is very probable that the student used 1.866181 to calculate the value of the definite integral, because the correct answer (106.109) rounds to two-places as 106.11. However, the student loses the answer point due to a decimal presentation error. The student only earns 1 point for writing the definite integral.

(6) Teachers should stress communication and notation. In part (a), we introduce the function $v_P(t)$. If a students drops the subscript in part (a) or in part (b), and writes only $v(t)$, they were forgiven, as there is only one function at this stage in the problem. However, in parts (c) and (d), we get another function called $v_Q(t)$. So in parts (c) and (d), the subscripts mattered. Simply writing "$v(t)$" is now ambiguous, and did not earn points. Worse was dropping "(t)" from the function. For instance, consider the following student solution for part (c), which is exactly like the one above, with one small change.

The interval is $[1.866, 3.52]$ and $\int_{1.866}^{3.52} v_Q t\,dt = 106.169$.

Here, the student earns no points at all: they do not get the interval point due to the decimal presentation error; they do not get the answer point since it is not the correct answer; and they lose the definite integral point since it now looks like they want to integrate v_Q times t. (If the "t" was absent entirely, they also did not earn the definite integral point.)

Practice using functions with subscripts with your students. Get them used to writing proper mathematical notation. It cannot be stressed enough that communication of mathematics is a fundamental aspect of the free-response problems on the AP Exam.