2018 BC-6 Dennis Wilson, Landmark Christian School

Problem Overview:
Students were given the Maclaurin series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

and were told that this series converges to $\ln(1+x)$ on its interval of convergence. The function f was then defined as $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

Part a:
Students were asked to write the first four non-zero terms and the general term of the Maclaurin series for f.

Part b:
Students were asked to determine the interval of convergence for the Maclaurin series of f, showing the work that leads to their answer.

Part c:
Students were asked to use the alternating series error bound to find an upper bound of $|P_4(2) - f(2)|$, where $P_4(x)$ is the fourth degree Taylor polynomial for f about $x = 0$.

General scoring guidelines for the problems:

Part a: (2 points)

The first point was awarded for the first four terms. To earn this point, the terms must be correct with the x term distributed throughout each. The terms did not need to be simplified algebraically nor did they need to be summed. The terms could simply be listed.

The second point was earned with the correct general term. This term did not need to be simplified either. If the series was presented with summation notation, the reader could ignore indices. The parity of the powers of -1 and x must be the same. The general term must be presented in part (a) to earn this point. A correct general term presented later in the problem would not earn it.

While students who answered part (a) with incorrect terms earned no points, their incorrect general term could be imported into later parts to earn points under certain conditions.

Part b: (5 points)

With five points available for part (b), readers were encouraged to score this part last. The first point was earned with a correct ratio of the next term to the current term, $\frac{a_{n+1}}{a_n}$. Neither the presence of a limit nor the absolute value was necessary for the point. The “alternator” or "toggle", $(-1)^n$, could also be missing. The student could also first present the ratio as an appropriate product and earn this point.

The second point was earned for the correct limit. This required an explicit indication of a limit but not the absolute value. Students who incorrectly linked the limit with an indeterminate ratio using an equal sign $\left(= \frac{\infty}{\infty}\right)$ could not earn this point.

Students earned the third point for the radius of convergence. Students who presented the correct general term in part (a) needed the correct ratio and a nontrivial of radius of convergence to earn the point. Therefore, students who incorrectly computed the limit could still earn this third point provided that their limit did not create a trivial calculation for the radius.

Students needed to consider both endpoints for the fourth point. To be eligible, students must have an interval of convergence centered at $x = 0$ with a finite, nonzero radius. "Considers" does not simply mean "identifies." The student must substitute their endpoint into a consistent general term for the endpoint to be considered.

Correct justification for both endpoints with a correct interval $(-3 < x \leq 3)$ earned the fifth point. The student must state that series diverges at $x = -3$ with any one of various correct reasons. The simplest way to justify their answer was to state that the series was a multiple of the harmonic series. The student must also state that the series converges at $x = 3$ with correct reason. The simplest reason was to state that the series is a multiple of the alternating harmonic series.

Incorrect general terms could be imported into part (b). In most of these cases, the student could earn three of the five points. The points were earned if the imported series did not trivialize the computations for each of five pieces of this part.

<u>Part c:</u> (2 points)

Students earned the first point in this part for using the first omitted term after the fourth-degree term given in their answer for part (a). If that answer was correct, then this would be a fifth-degree term. Some students gave a series with even symmetry (all even powers of x) in part (a). In this case a sixth-degree term was needed.

The second point was earned by the answer. A bald answer could not earn this point – evidence of the first omitted term must be present with the value of $x = 2$ substituted into this term. The presence of simply $\frac{2^5}{3 \cdot 3^4}$ would earn both points for this part.

<u>Observations and recommendations for teachers:</u>

(1) Students should have practice creating the Maclaurin series for functions from the series for another function. These series should require more than one operation and students should learn the correct order for performing each step. Many students multiplied the given series by x before performing the substitution. This not only lost both points for part (a) but also caused the

student to forfeit at least two points in part (b). Some students attempted to calculate the terms of the Maclaurin series from the respective derivatives of the function. These derivatives were complicated and time consuming and very few students were able to calculate their values correctly.

(2) While algebraic simplification is not necessary, and frequently discouraged on free response questions, students who simplified the powers of x for the first four terms and the general term usually earned more points. Students who did not simplify sometimes set up their ratio in part (b) incorrectly and most times chose the wrong term when finding their error bound in part (c).

(3) Students need to read the question carefully and note any functions that are named. The function $f(x)$ was defined in the stem of the problem. In part (b) the function was only referred to by the letter f. Many students erroneously used the series given for $\ln(1+x)$ in part (b) instead of the one they found in part (a).

(4) While students could earn various points in part (b) with invisible absolute values and limits, the students who used correct notation throughout the problem generally scored better than those who did not. Students should be given ample practice calculating the radius of convergence for a Maclaurin series from a given general term.

(5) Many students failed to earn any point in part (c) because they were confused by the meaning of “fourth-degree” Taylor polynomial. Students incorrectly used the Taylor polynomial with four terms (fifth degree) instead.