Problem Overview

A boat at sea is gathering data on plankton. The density of plankton is given as $p(h) = 0.2h^2 e^{-0.0025h^2}$, where h is the depth in meters for $0 \le h \le 30$, and p is in millions of cells per cubic meter. For $h \ge 30$, the plankton density is measured by the continuous function $f(h)$, which is not given.

Part a

Students were asked to compute $p'(25)$ and explain its meaning in context.

Part b

Students were asked to compute the number of plankton cells in a column of water from $h = 0$ to $h = 30$ where the column has square cross sections of area 3.

Part c

Students were told about a function $u(h)$ with two properties: $0 \le f(h) \le u(h)$ for all $h \ge 30$, and $\int_{30}^{\infty} u(h)\,dh = 105$. Assuming the column of water is $K > 30$ meters deep, students were asked to write an integral expression which gives the number of plankton cells, and to explain why the number of cells must be less than or equal to 2000 million.

Part d

Students were asked to find the distance traveled by the boat for $0 \le t \le 1$ with position $(x(t), y(t))$ where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$.

Comments on Student Responses and Scoring Guidelines

Part a

For part (a), the student could earn two points. There was 1 point earned for the correct value of -1.179 for $p'(25)$. No work needed to be shown for this since this is a calculator active problem. There was 1 point earned for the correct explanation with units. To earn the explanation point the student needed to satisfy four criteria: indicate that the value is a rate; mention density of plankton (not the number of plankton); indicate that the rate was for $h = 25$; and indicate the correct units of millions of cells per cubic meter per meter. The units could be expressed as "millions of cells/cubic meter/meter" or "millions of cells/m^4" or in other equivalent ways. If any of these four criteria were missing, the student did not earn the explanation point.

Part b

The student could earn two points for this part as well. Readers needed to see an integrand of $p(h)$ in a definite integral with a lower limit of zero to earn the first point. A correct evaluation of $\int_0^{30} 3p(h)\,dh = 1675$ earns the second point. No work needed to be shown for this definite integral since this was a calculator active problem.

Part c

There were three points available for the student to earn in part (c). To earn the first point, the student had to write an expression for the total amount of plankton cells as

$$\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh.$$

If the student did not use K appropriately, used ∞ instead of K, or used $u(h)$ instead of $f(h)$, the student did not earn this point.

To earn the other two points, the student had to indicate a comparison between the definite integral of f and the improper integral of u, and then use that to obtain a bound on the number of plankton cells which was less than 2000 million. In particular, to earn the second point, the student had to indicate that $\int_{30}^{K} f(h)\,dh$ is less than $\int_{30}^{\infty} u(h)\,dh = 105$. This could have been demonstrated in a variety of ways, from an inequality to a written description, but the use of a limit or the notion of convergence required careful handling by the student to earn the point. To earn the third point, the student had to evaluate the definite integral of $p(h)$ (which could be used from part (b)), and add $3 \times 105 = 315$ to it to get an answer less than 2000. If the student did not include the constant 3 in either the integral of $p(h)$ or the upper limit of 105, the student did not earn the point. A minimal response which earns all three points is the following:

$$\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh \le 1675 + 3(105) = 1990 < 2000.$$

However, the following similar response earned none of the three points:

$$\int_0^{30} 3p(h)\,dh + \int_{30}^{\infty} 3u(h)\,dh = 1675 + 315 = 1990 < 2000.$$

There are two reasons this earned none of the three points. The first was that this response does not mention the function f nor the constant K (that costs the student the first two points). The second is that the number 1990 was never indicated as a bound (and that costs the student the third point); that is, 1990 is not declared as the maximum possible but the actual number of plankton.

Part d

Two points could have been earned for this problem. The first point was for the set up, and the second for the evaluation. A correct response earning both points is

$$\int_0^{1} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = 757.455.$$

Students had to use notation appropriately to earn the set up point; missing parentheses, prime marks, or squares did not earn this point, but could still earn the answer point (with some exceptions; see Observation #5 below). If the notation was especially egregious, the student lost both points.

Observations and Recommendations for Teachers

(1) Students should learn how to use their calculators. Many students lost easy points for incorrect answers obtained from a calculator. It was also clear that many student calculators were in degree mode for part (d), and did not earn the point. There were numerous instances of students actually finding an expression for $p'(h)$ and then plugging in $h = 25$. This is a waste of valuable time since this is a calculator active problems. (Some students also did not find the derivative correctly and lost the answer point.)

(2) Students should read the problem. In part (a), correct numerical answers were presented with lacking contextual explanations, or no explanation at all. Many students lost the explanation point by writing

> The value $p'(25)$ means that the rate of change of the density of plankton cells is decreasing at a rate of -1.179 million cells per cubic meter per meter when $h = 25$.

This repsonse has all the correct criteria, but the students loses the point because the student indicates that something is decreasing at a negative rate, which means that something is increasing.

(3) Students should read the problem. In part (b), many students saw "cross sections" and assumed this was a volume of a solid of revolution problem, or that the cross sections were semicircles, or that the "area of 3 square meters" meant the the side length of the square cross sections was 3. Some students set the problem up correctly, but used "$A(x)$" instead of "$p(h)$" which indicates that the student was thinking about cross sectional areas, but as there is not function A in this problem, the student lost the integrand point.

(4) Students should carefully read the problem. There was a cornucopia of functions the student had to handle in part (c) — $f(h)$, $u(h)$, and $p(h)$, not to mention the constant K. Which functions were used in which context mattered. Not referring to the function f at all in part (c) automatically cost the student the first two points, and made it difficult to then earn the answer point. When there is only one function in a problem, called, for instance, g, and the student writes f, readers can sometimes assume the student simply misspelled "f", and the student may not lose any points for the mistake. However, in this problem, with functions p, u, and f, the presence of some other "mispelling" is disastrous: if the student writes $v(h)$, which function is the student really using?

(5) Teachers should stress communication and notation. Students who played fast and loose with either did not earn points for something in their work. Indeed, a common response to part (b) was

$$\int_0^{30} p(h)\, dh = 558.471645 \cdot 3 = 1675.$$

Although the integrand and answer are correct, the integrand as written is not equal to the given final answer, and the student did not earn the answer point. Another example is part (d). The correct expression for the distance traveled is

$$\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

However, students lost the set up point for missing a single (open or closing) parenthesis, or for miscopied coefficients (if they wrote the actual equations). If multiple parentheses were missing, or a square, or a prime mark, students lost both points, even if the correct numerical answer was given. A great example of this is the following response, which was seen many times:

$$\int_0^1 \sqrt{(x'(t))^2 + (y'(t)^2}\, dt.$$

In the second term, there is ambiguity in what is being squared—is $y'(t)$ squared, or just t squared? This one parenthetical mistake cost the student the set-up point, but we could still read for the answer. However, the following parenthetical mistake cost the student both points:

$$\int_0^1 \sqrt{(x'(t)^2 + (y'(t))^2}\, dt.$$

Since the second closing parenthesis on the first term is missing, but is present at the end of the second term, the expression now reads as $\sqrt{((x')^2 + y')^2}$, which simplifies to $(x')^2 + y'$. This is not the speed, and the student loses both points.