Problem Overview

The student is given a table indicating the height of a tree at various times, and that the height function is twice-differentiable.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

Part a

Students were asked to use the table to estimate $H'(6)$ and then to interpret its meaning in context of the problem with the correct units.

Part b

Students were asked explain why there is at least one time in the interval $2 < t < 10$ such that $H'(t) = 2$.

Part c

Students were asked to approximate the average height of the tree over the interval $2 < t < 10$ using a trapezoidal sum with subintervals as given in the table.

Part d

Students were told that the height could be modeled by the function $G(x) = \dfrac{100x}{1+x}$ where x is the diameter of the base of the tree in meters. Students were told that the diameter is increasing at 0.03 meters/year when the tree is 50 meters tall. Finally, students were asked to find the rate of change of the height with respect to time when the tree is 50 meters tall.

Comments on Student Responses and Scoring Guidelines

Part a

Students could earn two points in this part. Students earned the first point by using the difference quotient

$$\frac{H(7) - H(5)}{7 - 5}$$

to estimate $H'(6)$. No other difference quotient earned the point. Students earned the second point by indicating four things: that the numerical value was a rate; that the rate pertained to the height of the tree; that the value is an estimate for $t = 6$; and that the units are meters per year. Students could indicate these criteria in various ways, such as "$H'(6) = 2.5$ means the height is increasing at 2.5 m/y at the 6th year." Although "increasing" was not required, if a student wrote "decreasing" with a positive difference quotient, they did not earn the point. The student did not need to reference the numerical value of the difference quotient in the explanation to earn the second point, which means that the student with an incorrect value of the difference quotient could still earn the interpretation point.

Part b

This problem was about the Mean Value Theorem and two points could be earned. To earn the first point, students had to compute the difference quotient over the interval $3 < t < 5$ as this was the only interval which has an average rate of change of 2. To earn the second point, students had to show that the Mean Value Theorem could be applied by saying that H is a continuous function, and then writing the conclusion. Students did not need to say that H was differentiable since this was given in the problem. Many students did not earn the first point because they tried to use the interval $2 < t < 10$ in the difference quotient, and did not earn the second because they did not state that H was continuous. It was not necessary to mention the Mean Value Theorem by name, although that helped the readers know the student's intent.

An argument using the Intermediate Value Theorem could have been used to earn both points. Such an argument went somthing like this.

> Since both H and H' are continuous, and
>
> $$\frac{H(3) - H(2)}{5 - 2} = \frac{1}{2} \qquad \text{and} \qquad \frac{H(7) - H(5)}{7 - 5} = \frac{5}{2},$$
>
> then by the IVT, there must be a point c in the interval $2.5 \le t \le 6$ such that $H'(c) = 2$.

Now, the slope calculations would earn the first point (and the average rate of change on $5 < t < 7$ could be merely referenced if already calculated in part (a)). These calculations are valid because of the Mean Value Theorem, and *that* requires stating that H is continuous, as in the MVT argument described above. To complete the argument, and earn the second point, students had to apply the IVT to H'. The IVT requires the continuity of H', which had to be explicitly stated. The fact that $\frac{1}{2} < 2 < \frac{5}{2}$ guarantees a value t such that $2 < t < 7$ and $H'(t) = 2$. Neither theorem had to be named, although if an incorrect name was used, students did not earn the second point. (Most students who attempted the IVT argument did not earn both points. The most common mistake was neglecting to cite the continuity of H'. Indeed, in the over 1000 AB4 papers this Reader scored, there were 2 students who earned both points using this argument.)

Part c

Two points were available for this part. One point was earned for a correct sum of products which resulted in the correct trapezoidal areas. One point was earned for the average value. Simplification of arithmetic was not necessary. A response such as

$$\frac{1}{8}\left(1\cdot\frac{1.5+2}{2}+2\cdot\frac{2+6}{2}+2\cdot\frac{6+11}{2}+3\cdot\frac{11+15}{2}\right)$$

or

$$\frac{1}{8}\left(\frac{3.5}{2}+8+17+\frac{3\cdot 26}{2}\right)$$

earned both points. However, the response

$$1.75+8+17+39$$

lost both points as this was not an average value and no products were indicated. Some students realized that trapezoidal sums are equivalent to the average of the left- and right-hand Riemann sums, which also earned the trapezoidal area point. Graphs and drawings of trapezoids by themselves were not accepted; we still needed to see a clear sum of products.

Part d

To earn all three points, the student had to find the implicit derivative dG/dt by using the quotient rule (1 point) and the chain rule (1 point) and then evaluate this derivative when $G = 50$ (1 point). The student could rewrite $G(x)$ as $G(x) = 100x(1+x)^{-1}$ and use the product rule and chain rule instead. If the chain rule was not used at all, the student could only earn the one quotient rule point, and to earn that one point there could be no mistakes made in using the quotient rule (or product rule). In order to evaluate the derivative, the student must have calculated $x = 1$ from the information $G(x) = 50$ and used this x-value in their derivative.

Observations and Recommendations for Teachers

(1) Students should know how to read a table. In parts (a), (b), and (c), we read student work where they were mistakenly interchanging H-values with t-values in their calculations. Students, when asked to estimate $H'(6)$, seemingly believed that 6 was between 7 and 10 as evidenced by their difference quotient over the interval $7 < t < 10$. Students also misread the units in the table: some reported the units of $H'(6)$ as meters per year per year, believing that H was a rate. Students should be given plenty practice with tabular problems, and with the tables organized in different fashions with both function values and derivative values. Table problems are not going away.

(2) Students should avoid reciting formulas. Many students simply recited the Mean Value Therorem without ever connecting it to the table, the function H, or a difference quotient. Students who did this earned no points for part (b). This also got them in trouble in part (c), as many students tried to use the Trapezoid Rule, which is not applicable since the subintervals in the table are not equal. This also earned the student no points in part (c). The Trapezoid Rule is not part of the AP Calculus Curriculum Framework because we want to ensure the students understand what they are doing, not simply reciting a formula. (Yes, really, there is no Trapezoid Rule in the AP curriculum, nor was it in there before the re-design. However,

students *are* required to use trapezoids to approximate areas. You don't believe me? Go look for "trapezoid rule" in the Curriculum Framework. I'll wait.)

(3) Students should be prepared to use the major theorems of calculus. Although an argument using the Intermediate Value Theorem was accepted for part (b), the Mean Value Theorem was the intended solution. Students had to "jump through more hoops" to earn both points using the IVT, whereas using the MVT is less work. Part of bring prepared is teaching the students to be on the look out for particular set-ups: When asked to justify whether a *derivative* has a certain value on an interval, the student should consider the MVT; when asked to justify whether a *function* has a certain value on a interval, the student should consider the IVT.

(4) Students should read the problem. A remarkably small number of students earned both points in part (c) because most did not find the average value of $H(t)$. A remarkably small number of students earned the third point in part (d) because many never obtained the value $x = 1$ from $G(x) = 50$. Either students thought 50 was a diameter or 0.03 was a diameter, and used one of those two values as the value of x. Careful reading of the problem is a must, and teachers must train their students to read carefully.

(5) Students should avoid arithmetic. A little more than half of the students did the arithmetic in part (c), which was not necessary. The correct arithmetic leads to the fraction $263/32$, but some students felt they needed to report a decimal. They dutifully did the long division to obtain a decimal… which they did not provide to three decimal places. Even on the non-calculator section, the three-decimal-place rule is enforced. Between the arithmetic errors and the decimal place rule, most of the students who attempted the arithmetic did not earn the answer point.

(6) Students should "attend to precision". In part (d), the vast majority of students used the quotient rule right away (or rewrote the problem and used the product rule). However, a large number of students made sign errors, parentheses errors, chain rule errors, and calculation errors. Quotient rule attempts such as the following were unfortunately common, and did not earn the quotient rule point:

$$\frac{(1+x)100+100x}{(1+x)^2}, \qquad \frac{1+x\cdot 100-100x}{(1+x)^2}, \qquad \frac{(1+x)100-100x}{1+x^2}, \quad \text{and} \quad \frac{100x-(1+x)100}{(1+x)^2}.$$

Combine these with chain rule, and we have responses which did not earn two points, solely due to lack of attention to detail. Responses such as these, for example, did not earn either the quotient rule or chain rule points:

$$\frac{1+x(100)-100x\cdot\frac{dx}{dt}}{(1+x)^2} \quad \text{and} \quad \frac{(1+x)100\frac{dx}{dt}-100x}{1+x^2}.$$

In some of these cases we could read along for the student's consistent answer, but if the mistake was with the chain rule, the student automatically did not earn the answer point. (Thus, the student who responded with either of the two samples above lost all *three* points.) Students should understand that notation matters because it affects the communication of their intended answer, and teachers should endeavor to model good notation for their students.