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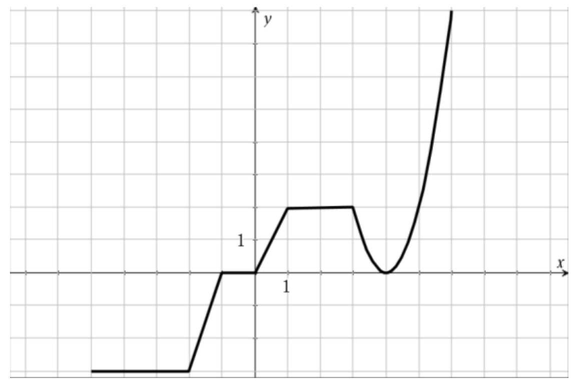
**Problem Overview:**
Students were given the graph of a continuous piecewise function $g$ which is the derivative of the function $f$. The function $g$ is piecewise linear for $-5 \leq x < 3$ and $g(x) = 2(x-4)^2$ for $3 \leq x \leq 6$.



Graph of $g$

**Part a:**
Students were asked to find the value of $f(-5)$ given that $f(1) = 3$.

**Part b:**
Students were asked to evaluate $\int_1^6 g(x)\, dx$.

**Part c:**
Students were asked to find the open intervals in $-5 < x < 6$ for which the graph of $f$ is both increasing and concave up. Students were told to give a reason for their answer.

**Part d:**
Students were asked to find the $x$-coordinate for each point of inflection of the graph of $f$, giving a reason for their answer.

**General scoring guidelines for the problems:**

The stem of the problem states that $g(x)$ is equivalent to $f'(x)$. This equivalence was accepted through the question and did not need to be explicitly stated by the student. Their equivalence will be understood throughout these guidelines and $g(x)$ will always be used for consistency though $f'(x)$ could also be used in its place.

**Part a:** **(2 points)**

The first point is awarded for the definite integral. The definite integral could be evaluated from $-5$ to $1$ or from $1$ to $-5$. The student could earn this first point without a definite integral if there was an arithmetic expression equivalent to the integral. The expression must have separate values for the area above and below the axis and they must be summed. For example,

$$-9 + \frac{3}{2} + 1 \qquad \text{or} \qquad -10\frac{1}{2} + 1$$

would earn this point. The reader could also look for evidence of the sum on the graph given in the stem of the problem.

The second point was earned for the answer of $\frac{25}{2}$ with supporting work, which may or may not have earned the first point. The minimal answer without an integral that would earn both of these points was $3-(-10\frac{1}{2}+1)$.

The point for the answer was often lost due to linkage errors from expressions similar to

$$\int_{1}^{-5} g(x)\,dx = \frac{19}{2} + 3 = \frac{25}{2}$$

Some students attempted to solve for $f(-5)$ analytically by finding an algebraic expression for $f(x)$. This method was abundant with time-consuming work which seldom resulted in a correct answer.

**<u>Part b:</u> (3 points)**

Students earned the first point in this part for splitting the definite integral at $x=3$. While the integral $\int_{1}^{3} g(x)\,dx$ could be split various ways or simply written as 4, the other part of the integral, $\int_{3}^{6} g(x)\,dx$ must be present to earn the point.

The second point was earned for the antiderivative of $2(x-4)^2$. This could occur in multiple ways: $\frac{2}{3}(x-4)^3$, $\frac{2}{3}u^3$ where $u=x-4$, or $\frac{2}{3}x^3-8x^2+32x$ if students expanded the function before finding the antiderivative. Students could earn the antiderivative point without an algebraic expression if the student presented work which contained two terms evaluated at both endpoints.

$$\frac{2}{3}(6-4)^3-\frac{2}{3}(3-4)^3$$

The final point for this part was earned for the numerical answer of 10. This answer could be a non-simplified arithmetic expression. This answer point could only be earned with the antiderivative. Some students failed to find the antiderivative of the function, instead immediately substituting the endpoints. They also arrived at an answer of 10. These students did not receive the answer point.

See more information in **<u>Observations and recommendations for teachers:</u>** (5.)

**<u>Part c:</u> (2 points)**

The first point in part (c) was earned for the correct intervals. The endpoints for these intervals could either be included or excluded, even though the question asked for open intervals. Students did not receive the point for the interval if they reported $0 \le x \le 1$ and $(4, b)$ with $b > 6$ since the domain of the function $f'(x)$ did not include values of $x$ greater than 6.

To be eligible for the second point, students must present either the correct intervals, the incorrect larger interval stated above, or exactly one of the correct intervals and a subset of the other. If one of these answers were presented, then students could earn the second point for the correct reasons. If students chose to use $f''$ in their reasoning, they were required to have explicitly stated $g' = f''$.

Many students did not earn the reasoning point because of references to either the function, slope, graph, or derivative without designating to which function they were referring.

There was a special case for part (c). If a student gave the intervals $(0,3)$ and $(4,6)$ with the reason that both $f$ and $f'$ are increasing, then both points were earned. (For explanation of this case see **<u>Observations and recommendations for teachers:</u>** (7) below.)

**<u>Part d:</u> (2 points)**

Students earned the first point in this part with a correct answer. These answers could vary but each answer must have an $x$-coordinate of 4. Bald answers were accepted for this point and could be $x = 4$, $\left(4,7\frac{2}{3}\right)$, or $x = 4$ and any subset of $[-5,-2] \cup [-1,0] \cup [1,3]$. Students did not earn the answer point if they gave the coordinates of $(4,0)$ which is a point on the graph of $f'$ and not $f$.

To earn the second point for the reason, students must have earned the answer point or given $(4,0)$ as the point of inflection. There were two different cases for how the reason point was awarded.

In the first case, if the answer given was $x = 4$, $\left(4,7\frac{2}{3}\right)$, or $(4,0)$ then students earned the point with a correct reason using $g$ and/or $g'$.
Some examples of appropriate reasons include:

$g$ changes from increasing to decreasing
$g'$ changes signs
$g'$ changes from negative to positive
The slope of $g$ changes from positive to negative or vice versa
$g$ has a local extremum

Once again, a student must make an explicit statement of $g' = f''$ to give reason using $f''$. This explicit statement could have come in part (d) or any previous part of the question.

The second case dealt with the answer of $x = 4$ and any subset of $[-5,-2] \cup [-1,0] \cup [1,3]$. To earn the reason point, students must give $x = 4$ and the entire interval. If just a subset was given, then the student was not eligible for the reason point. The only reason accepted for this second case was that $g$ changes from decreasing to increasing or increasing to decreasing.

**Observations and recommendations for teachers:**

(1) While students were not required to explicitly state $f' = g$, it is good practice for students to algebraically write the implications of the information given in the stem of the problem. In this case, that would also include an explicit statement of $f'' = g'$. These statements carry forward through all the parts of the problem, therefore the student should make these declarations in the stem. If the student were to make the statement $f'' = g'$ in part (d), the student would not receive the reason point in part (c) if using $f''$ for the reasoning.

(2) This is the first free response question since 2009 to contain the graph of a piecewise function for which the definite integral could not be evaluated completely with the geometry of the graph. It is the first time that the student was required to split the integral to evaluate the parts separately. It is recommended that students be given practice evaluating the definite integrals of piecewise functions which require this.

(3) The antiderivative in part (b) could be evaluated in many ways. Some students used $u$-substitution and some expanded the binomial. Both of these ways were time consuming and prone to introducing errors. Students should be presented with a third way of evaluating integrals that involve the composition of functions where the inside function is linear: $\int f'(mx+b)\,dx = \frac{1}{m}f(mx+b)$. Integrals of this type occur frequently, and this technique provides students with a quick and reliable method for evaluating them.

(4) It cannot be stated enough that students are not required to perform algebraic simplifications or arithmetic computations. If these operations are required in your class, students should know they are released of this requirement on the free response portion of the AP exam. These operations essentially do two things to students on the exam: they consume valuable time and they qualify the student for a chance to make a mistake which will forfeit the answer point.

(5) Correct mathematical methods are always accepted on the Free Response Questions unless a specific method is stated. For the definite integral in part (b), a method attributed to Archimedes was accepted for the determining the area bound by the parabola and the axis. This method states that this area is one-third of the area of the circumscribing rectangle. Students who used this method could earn all three points for part (b) with the expression $4 + \frac{1}{3}(1)(2) + \frac{1}{3}(2)(8)$.

(6) Communicating is a mathematical practice for AP Calculus in the new frameworks. There were very few pronouns used on the exams I read. Unfortunately, students continued to use vague phrases that act essentially the same way as pronouns. Words like graph, derivative, and slope have no meaning without reference to a named function. Students need to be precise when writing sentences to give reason for answers. Also, while nonmathematical terms like "turning point" may help students develop their conceptual understanding, the use of such terms will not earn points on the AP exam. Proper mathematical terminology should be reinforced as students develop their understanding of calculus throughout the school year.

(7) Reasoning with Definitions and Theorems is also a mathematical practice for AP Calculus in the new frameworks. Definitions are important, but they can vary from textbook to textbook. The definition for concavity given in most standard calculus textbooks usually states that the graph of a function is concave up on an interval if the derivative of the function is increasing on that interval. The special cases in parts (c) and (d) arose from the varying definitions for an increasing function. Introductory calculus texts usually define a function to be increasing if $x_2 > x_1$ implies $f(x_2) > f(x_1)$. More advanced textbooks will usually consider this to be the definition for strictly increasing. These texts define a function to be simply increasing if $x_2 > x_1$ implies $f(x_2) \geq f(x_1)$. If this definition is used, then the function $g$ is increasing on the intervals $[-5,3]$ and $[4,6]$. This consequently means that the function $f$ is concave up on these intervals. If a similar definition is used for decreasing, then the function $g$ is decreasing on the intervals $[-5,-2]$, $[-1,0]$, and $[1,4]$. Therefore $f$ is concave down on these intervals.

(8) Notice that these alternate definitions for increasing and decreasing lead to the fact that $f$ can be both concave up and concave down on the intervals where the graph of $g$ is horizontal. It is for this reason that any values of $x$ in these intervals were accepted for the $x$-coordinates of the points of inflection. At any of these points, the graph of $g$ changes from an interval of increasing to decreasing or vice versa under these definitions. It is suggested though that teachers use the first set of definition for increasing and decreasing $(x_2 > x_1 \rightarrow f(x_2) > f(x_1))$. This yields the answer stated in the scoring guidelines and removes ambiguity in understanding the point of inflection.