Problem Overview:

Let f be the function defined by $f(x) = e^x \cos(x)$.

Part a:

Students were asked to find the average rate of change of f on the interval $0 \le x \le \pi$.

Part b:

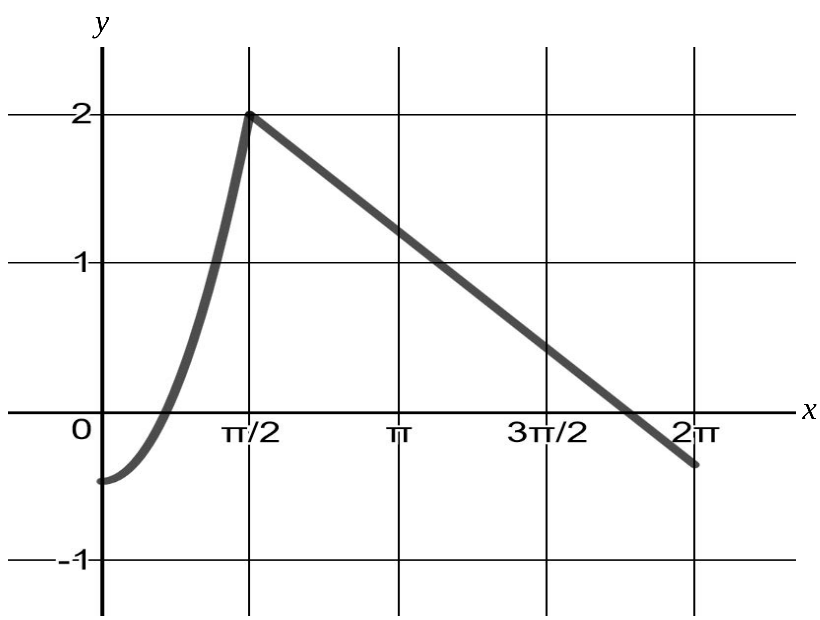
Students were asked "what is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?"

Part c:

Students were asked to find the absolute minimum value of f on the interval $0 \le x \le 2\pi$ and justify the answer.

Part d:

Students were given that the function g is differentiable and $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Students were asked to find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist and justify the answer.



Graph of g'

Comments on student responses and scoring guidelines:

Part a:

This part was worth one point for the answer $\frac{-e^{\pi}-1}{\pi}$. If supporting work was shown, readers had to look carefully because, for example, $\frac{f(\pi)-f(0)}{\pi}=\frac{f'(\pi)-f'(0)}{\pi}$ for this f. The second form of the difference quotient shown as supporting work would not earn this point because the correct answer was computed incorrectly. Some students recognized that the average rate of change is the average value of $f'(x)$ over the interval and could be calculated using $\frac{1}{\pi-0}\int_0^{\pi} f'(t)\,dt$. A few students incorrectly tried calculating $\frac{1}{\pi-0}\int_0^{\pi} f(t)\,dt$. Simplification is not required meaning that $\frac{e^{\pi}\cos(\pi)-e^{0}\cos(0)}{\pi-0}$ earned this one point.

Part b:

The first of two points was awarded for a correct expression for f' such as $e^{x}(-\sin(x))+e^{x}\cos(x)$. An ambiguous expression such as $e^{x}-\sin(x)+e^{x}\cos(x)$ had to be resolved in subsequent work. The second point was for the answer either simplified as in $e^{3\pi/2}$ or unsimplified as in $e^{3\pi/2}\left(\cos\left(3\pi/2\right)-\sin\left(3\pi/2\right)\right)$. In fact, the latter expression earned both points. In trying to show the product rule, students sometimes showed one of three forms of a sign error: $e^{x}\sin(x)+e^{x}\cos(x)$, $e^{x}\sin(x)-e^{x}\cos(x)$ or $-e^{x}\sin(x)-e^{x}\cos(x)$. These did not earn the first point but were eligible for the second point showing correctly $-e^{3\pi/2}$ in the first two cases or $e^{3\pi/2}$ in the last case. Other derivative errors such as $f'(x)=-\sin(x)e^{x}$ earned zero points.

Part c:

Three points could be earned, the first for setting f' equal to zero. This first point could be earned in a variety of ways such as $f'(x)=0$ or stating "f' changes sign…" Students are looking for an absolute minimum value on a closed interval. Thus the interior points where f' is 0 must be examined. The second point was for finding the two values of x for which this was true. The correct derivative or the sign error derivative $e^{x}\sin(x)-e^{x}\cos(x)$ resulted in $x=\frac{\pi}{4},\frac{5\pi}{4}$. The other sign error forms of the derivative resulted in $x=\frac{3\pi}{4},\frac{7\pi}{4}$ and in the absence of other, incorrect, values of x students showing these values earned the second point. The third, justification, point could only be earned using the correct derivative and x values. The best way to do this on a closed interval is to show the function values at the endpoints and the interior critical points, followed by stating the minimum value of the function. This display of four function values eliminates the value at the point where $x=\frac{\pi}{4}$ from consideration. Some students tried a local argument regarding the sign change in f' where $x=\frac{\pi}{4}$ and did not earn the third point using that argument.

Part d:

The first of three points in this part of the problem was awarded for stating that g is continuous (see **Observations and recommendations for teachers** (4) below) and displaying $\lim_{x\to\pi/2} f(x)=0$ and $\lim_{x\to\pi/2} g(x)=0$ (see **Observations and recommendations for teachers** (5) below). The second point was earned for applying L'Hopital's rule as in $\lim_{x\to\pi/2}\frac{f'(x)}{g'(x)}$ or $\lim_{x\to\pi/2}\frac{f'(x)}{2}$ where limit notation is attached to a ratio of derivatives (see **Observations and recommendations for teachers** (6) below). Work such as $\lim_{x\to\pi/2}\frac{f(x)}{g(x)}=\frac{f'(x)}{g'(x)}$ did not earn the second point because of missing limit notation. Use of explicit notation such as $\lim_{x\to\pi/2}\frac{e^x(-\sin(x))+e^x\cos(x)}{2}$ also earned the second point. Importing any bad derivative of f from part b or c could earn this second point. Any incorrect sign error derivative mentioned in **Part b** above was eligible for the third point in this part of the problem. Work such as $\lim_{x\to\pi/2}\frac{e^x\sin(x)+e^x\cos(x)}{2}=\frac{e^{\pi/2}}{2}$ earned both the second and third points despite the fact that this final answer is incorrect; it is, however, consistent with this form of a bad derivative. The presence of $\frac{0}{0}$ in student work was treated as scratch work or just thinking about the problem. However, explicitly linking $\lim_{x\to\pi/2}\frac{f(x)}{g(x)}=\frac{0}{0}$ lost students the first point.

Observations and recommendations for teachers:

(1) The average rate of change of a function on a closed interval is the ratio of the change in function values at the endpoints of the interval to the length of the interval. This is an important difference quotient that leads to the calculation of a derivative, but is not a calculus calculation. It is simply the slope of the line containing the endpoints of the closed interval.

(2) Calculating the slope of a line tangent to a curve at a point is one of the very first applications of the derivative. Calculation of the derivative is basic work to show in answering the question in part b. Students often went on to build an equation of this tangent line. Students should carefully read and answer the question that is asked. On the 2018 exam, a correct equation was accepted for the second point showing the value of the slope; but an incorrect equation, even with the correct slope, was not. There is no guarantee that this will be accepted in the future, because the answer to the question as posed is *not* an equation of a line.

(3) Once again on the AP Calculus Exam, students were asked to find the value of an absolute extremum of a function on a closed interval. This can best be done by finding the location of critical points on the interval and then calculating the values of the function at the critical points and endpoints. An absolute max or min can be determined by inspecting these values. Many students do not use this straight forward method, opting instead to use a local argument involving a change in sign of the derivative of the function. Don't do that! First of all, the endpoint values must be examined. Secondly, it is very difficult to phrase the local argument in such a manner that it eliminates all other possible values or guarantees the absolute max or min nature of the critical point. The closed interval argument, the local (first derivative test) argument, and the second derivative test are three different methods. They should be practiced separately. Later, teachers can compare these methods when it might be possible to use more than one method to answer a question about max or min values.

(4) In keeping with the emphasis in the curriculum on reasoning from definitions and theorems, L'Hopital's rule should be applied, but the hypotheses needed to be verified, for the first point in part d. Readers had to see clearly that the student knew both $\lim_{x\to\pi/2} f(x)=0$ and $\lim_{x\to\pi/2} g(x)=0$. Both functions f and g have to be differentiable. The differentiability of g is a given fact. Students did not have to state that f was differentiable, it being assumed from the explicit expression of f as a product of differentiable functions. However, to state that $\lim_{x\to\pi/2} g(x)=0$ when the only value of g that is known is $g\left(\frac{\pi}{2}\right)=0$ requires the observation that g is continuous. The definition of continuity at the point where $x=\frac{\pi}{2}$ is that $\lim_{x\to\pi/2} g(x)=g\left(\frac{\pi}{2}\right)$ which is why we know that limit has a value of 0. *Very* few students included reference to the differentiability or continuity of g in order to earn the first point in part d.

(5) Most calculus texts do not explicitly write, for L'Hopital's rule, that both $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ in order for the rule to be applied in evaluating $\lim_{x\to a}\frac{f(x)}{g(x)}$. This explicit writing is common in more advanced calculus or analysis texts but not in standard calculus texts covering the equivalent of Calculus I and II at the college level. Rare exceptions are *Calculus, Dynamic Mathematics* Volume I by C. Garner and *Calculus* by Stewart. The AP Calculus Exam has been requiring the statements $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ for a number of years going back to when this was a BC topic and not an AB topic……

(see 2013 BC5 part a and 2016 BC4 part c as examples). AP students need to be explicit in stating these limits before applying the rule. It is possible that one or both of these limits might require the stating of additional information or showing a computation.

(6) A few calculus texts show a "weaker" form of L'Hopital's rule:

If f and g are differentiable, $f(a) = g(a) = 0$ and $g'(a) \neq 0$ then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

The functions in part d satisfy these hypotheses. Limit notation does not need to be attached to a ratio of derivatives when showing work in this manner. It is recommended that students *do* show work attaching limit notation to a ratio of derivatives as in $\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \pi/2} \dfrac{f'(x)}{g'(x)} = \dfrac{f'(\pi/2)}{g'(\pi/2)}$. At the 2018 reading, showing limit notation attached to a ratio of derivatives was required for the second and third points. Stating that $\lim\limits_{x \to \pi/2} f(x) = \lim\limits_{x \to \pi/2} g(x) = 0$ was required for the first point, along with g being continuous at $\frac{\pi}{2}$.

(7) In keeping with increased emphasis on MPAC's "Communication" and "Reasoning with Definitions and Theorems," hypotheses of theorems must be shown before applying a theorem or a method such as L'Hopital's rule. Specifically writing $\lim\limits_{x \to \pi/2} f(x) = 0$ and $\lim\limits_{x \to \pi/2} g(x) = 0$ was required on the 2018 exam. It is never correct to write $\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$. That will cost students a point because of the specific linkage of the limit of the ratio of functions to an indeterminate "form."

(8) Again regarding L'Hopital's rule, the simple statement that $\lim\limits_{x \to a} f(x) = 0$ has required no work to be shown if the function is given explicitly. If a more involved calculation must be shown, or as in the case of AB5 part d, the continuity of the function must be reported in order for the limit to be known, then readers will expect more from students. One way for students to plan for this is to "write down things that are true." For example, if f is differentiable, then write that "f is continuous" before starting work. If g is twice differentiable, then write that "both g and g' are continuous." Avoid the temptation to proceed immediately into the solution work. These types of statements can also help students in establishing hypotheses when applying the Mean Value Theorem, the Intermediate Value Theorem, or the Extreme Value Theorem. Note that part c of this problem is an application of the EVT for which continuity of the function is required, although not as yet required to be stated by students on the exam.