

**Problem Overview**

A 10-ft tall tank has horizontal cross sections at height  $h$  defined by the continuous, decreasing function  $A(h)$ , where  $A$  is in square feet. The student is given a table of values of  $A$ , indicating three subintervals.

|                         |      |      |     |     |
|-------------------------|------|------|-----|-----|
| $h$<br>(feet)           | 0    | 2    | 5   | 10  |
| $A(h)$<br>(square feet) | 50.3 | 14.4 | 6.5 | 2.9 |

**Part a**

Students were asked to compute a left-hand Riemann sum using the values and the subintervals in the table to approximate the volume of the tank. Students were asked to give units of measure.

**Part b**

Students were asked to determine whether the Riemann sum was an overestimate or underestimate, and to explain why.

**Part c**

Students were given a model for  $A$  in the form of the function  $f(h) = \frac{50.3}{e^{0.2h} + h}$  and asked to find the volume of the tank from this model. Again, units of measure were necessary.

**Part d**

Students were to use the model from part (c) to determine the rate of change of the volume when the water is at a height of 5 feet, given that the height is increasing at 0.26 ft/min. Yet again, units of measure were required.

**Comments on Student Responses and Scoring Guidelines**

There was 1 point earned for units of measure in parts (a), (c), and (d). Students must have the correct units in all three parts and at least one of these units must be attached to a number to earn the point.

### **Part a**

Readers needed to see a sum of products using values from the table to earn the first point. At least 5 of the 6 numbers used in the sum of the products must be correct to earn the point. The students who wrote only  $100.6 + 43.2 + 32.5$ , or wrote these numbers inside rectangles drawn under a graph, did not show a sum of products and so did not earn the first point, but were still eligible for the second point. To earn the second point, the student could have left an unsimplified sum of products, simplified the sum of products partway, or simplified it completely resulting in 176.3.

### **Part b**

To earn the point, the student had to indicate that the approximation in part (a) was an overestimate, and base that conclusion on the fact that we were told  $A$  is decreasing. Any justification based on the derivative of  $A$  was not accepted and the student did not earn the point (because we were not told that  $A$  was differentiable, only continuous). However, the student who computed a right Riemann sum correctly in part (a), and then said the result was an underestimate because  $A$  is decreasing, did earn the point. Students who left part (a) blank and responded to this appropriately earned the point.

### **Part c**

For the student to earn the first point, readers needed to see the correct integrand and the correct limits. Nonzero constant factors in front of the correct definite integral still earned the point (they are considered copy errors). To get the second point, the student needed to have 101.325 on their paper attached to the correct definite integral or to the indefinite integral of  $f(h)$ .

### **Part d**

To earn both points for  $dV/dt$ , students had to demonstrate the chain rule and the Fundamental Theorem; demonstrating only one of these earned one of the first two points. Students were allowed to demonstrate this in many ways; each of the following earned both points:

$$\frac{50.3}{e-5} \cdot 0.26, \quad f(5) \cdot 0.26, \quad \text{and} \quad \frac{dV}{dh} \cdot \frac{dh}{dt}.$$

There was no penalty if the student mistakenly used  $A$  instead of  $f$ . The student writing

$$\frac{dV}{dh} = \frac{dV}{dh} \cdot \frac{dh}{dt} \text{ without any more work, or } \frac{dV}{dt} = f(10) \frac{dh}{dt}$$

earned one point for showing the chain rule, but did not earn the FTC point. Writing

$$\frac{dV}{dh} = f(h), \quad dV = A(h) dh, \quad \text{or} \quad \frac{dV}{dh} = f(5)$$

without any other work, earned one point for showing the FTC, but did not earn the chain rule point. Many students, instead of starting from defining the volume as  $V(h) = \int_0^h f(x) dx$ , started from defining volume

as  $V = f(h) \cdot h$ . Starting this way, students used the product rule to find  $dV/dt$ , which did not earn either point. To earn the third point, the student must have the answer 1.694 or the expression  $\frac{50.3}{e-5} \cdot 0.26$  or  $6.517 \cdot 0.26$ . Many students did not earn the answer point due to rounding error (one can only assume that since the given rate of 0.26 was to two decimal places, students thought that the answer should also be two decimal places). However, the student who confused  $A$  instead of  $f$  here did not earn the answer point if the student used the value of  $A(5)$  from the table to compute the answer. The student writing  $f'(5) = -1.303$  and declaring this as the answer did not earn any of the three points.

### **Observations and Recommendations for Teachers**

(1) Students should read the problem. When asked to indicate units of measure in three parts, students should be aware that this is most likely a point (or two), and it is an easy point to receive.

(2) Students should read the problem. In part (a), perfectly correct *right* Riemann sums did not earn either point. Students should answer the question they were asked and not the question they thought they were asked.

(3) Students should not simplify. An answer to part (a) that was only “ $2 \cdot 50.3 + 3 \cdot 14.4 + 5 \cdot 6.5$ ” earned both points. While easy to do by hand or with a calculator, students should not simplify this expression. Readers want to see that the student knows how to set-up the requested Riemann sum. Teachers should have students practice Riemann sum questions such as part (a) without calculators, so students break the habit of reaching for the calculator to do simplification of arithmetic. This calculator use can get students in trouble: bald (correct) answers of 176.3 did not earn either point.

(4) Students should read the problem. In part (b), many students did not earn the point because they assumed every function given on a calculus test must be differentiable. But students can only use information they have been explicitly told, or information they can justify based on the information told. That  $A$  is differentiable is not one of them.

(5) Students should read the problem. Many students thought part (c) was a solid of revolution problem, and therefore computed  $\pi \int_0^{10} (f(h))^2 dh$ . Most likely, students thought this because they were trained to see “key words” like *volume* and they responded accordingly. Instead, students should understand the circumstances under which a definite integral can represent the volume of a solid.

(6) Students should read the problem. There was a flurry of units in part (d), but the astute student could do really well on this part if the student carefully read for the units to make sense of the rate they were to compute. Many students left part (d) blank; this actually cost the student *four* points because they were ineligible for the units point with a blank part (d).