2016 BC6 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were told that the function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. Further information about f was provided: $f(1)=1$, $f'(1)=-\frac{1}{2}$ and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1)=(-1)^n\dfrac{(n-1)!}{2^n}$.

Part a:

Students had to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

Part b:

Students had to show work in finding the interval of convergence of the Taylor series about $x = 1$ given that the radius of convergence is 2.

Part c:

Students were told that the Taylor series about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Students were asked to use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

Part d:

Students had to show that the approximation in part (c) is within 0.001 of the exact value of $f(1.2)$.

Comments on student responses and scoring guidelines:

Part a:

The first point was for the first two terms. The second and third points were for the third and fourth terms. The fourth point was for the general term. The first two terms were easily spotted, either correct or incorrect. The third and fourth terms were written by students in a variety of manners. Some third and fourth terms which are correct are: third term: $\frac{1}{8}(x-1)^2$ or $\frac{1}{2^2}\frac{(x-1)^2}{2}$; Fourth term: $-\frac{1}{24}(x-1)^3$ or $\frac{-2!}{2^3}\frac{(x-1)^3}{3!}$.
The general term could be presented or be part of a summation, in which case the index was ignored. Some correct versions of this are $\frac{(-1)^n}{2^n n}(x-1)^n$ or $\frac{(-1)^n}{n}\left(\frac{x-1}{2}\right)^n$ or $\frac{(-1)^{n-1}}{2^{n-1}(n-1)}(x-1)^{n-1}$.

Part b:

The first point was for the endpoints of the interval of convergence, which could have been seen in student work in interval notation such as [–1, 3]. Some students, not realizing that a given center and radius of convergence located the endpoints, worked with a ratio test. This work was ignored. The second point was for examining the endpoints, showing work to determine convergence or divergence, AND giving the actual interval of convergence which is (–1, 3]. A minimal approach could simplify and state something like the following: $x = -1 \ \frac{1}{n}$ and $x = 3 \ \frac{(-1)^n}{n} \rightarrow (-1, 3]$ or a minimal approach could "identify" as follows: $x = -1$ harmonic $x = 3$ alt harmonic $\rightarrow (-1, 3]$. If student work showed both of these approaches, both had to be correct in order to earn the second point. Students showed various versions of a simplified series term at the endpoints, but even if correct, the "simplify" method had to state the type of series before the final answer was awarded the point unless the simplifications were presented as $\frac{1}{n}$ and $\frac{(-1)^n}{n}$. In other words, a simplification had to get all the way to the basics unless the series were named as in harmonic, alternating harmonic, or Leibniz series (alternating series test term used by a few students).

Part c:

Students had to show use of the first three terms in determining the approximation to $f(1.2)$. Possible answers in the presence of this work, valid for this one point, are 0.905, $\frac{7.24}{8}$, and $1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2$. If a student used incorrect terms from part (a), the terms had to be at least quadratic and of the form $c(x-1)^k$ with $c \neq 0$ in order to earn this one point.

Part d:

The first of two points in part (d) was for the error term. This could not show up initially in a completely simplified form. Thus, "Lagrange error… $\frac{(0.2)^3}{24}$" earned the first point, even though readers were always looking for the alternating series remainder test (AST). An error term using AST had to be the next term after the student work in part (c) and that series needed to be alternating and decreasing. The second point was for a connection to the error bound as in ≤ 0.001. Students did not need to name the test being used. A minimal answer worth two points is $\frac{(0.2)^3}{24} \leq 0.001$. If no work was shown (student proceeded directly to simplified form) zero points were awarded as in $\frac{1}{3000} \leq 0.001$

Observations and recommendations for teachers:

(1) Terms in a Taylor series involve $(x - center)^n$, the nth derivative of the function evaluated at the center, and division by $n!$. This should be practiced both in situations where the derivatives must be calculated and in situations where an expression or a value from a table gives the derivatives.

(2) An interval of convergence has a center and a radius. The radius is the distance left and right from the center, and that distance locates the endpoints. Knowledge of this fact would have saved students from trying the ratio test in this problem.

(3) The difficult part of finding an interval of convergence is examining the endpoints. Students need to substitute the endpoint into the *original* general term for the series, and determine the properties of the resulting series formed at each endpoint. Some students have difficulties with this because using the ratio test uses the absolute value of the general term, and absolute value is sometimes still used as students try to check endpoints. Familiarity with basic series such as harmonic, alternating harmonic, and *p*-series is essential. All should be familiar to students before finding the complete interval of convergence. One way to teach this and practice in the classroom is to find the interval of convergence, less endpoints, and practice that before learning convergence tests. It doesn't hurt to "almost" find intervals of convergence and revisit them after study of convergence tests.

(4) The AP exams frequently ask for approximations, sometimes from a tangent to a function (the first order Taylor polynomial) and sometimes from more than one term in a Taylor series. Historically, such approximations were important. We now have a calculator to give us a several decimal place approximation to such values as sin(2.3) but that was not always the case. All that has to be done after the requisite terms have been determined is to substitute that value (in the case of sin(2.3) just let $x = 2.3$).
Reminder: simplification of the arithmetic is not required on the AP Calculus exam.

(5) The Lagrange form of a remainder (error) is a required topic for BC AP Calculus. However, the use of the AST should be emphasized in class so that this (often easier) calculation of an error is well known to students. The AST is also a required topic for BC.

(6) Since the Lagrange and AST are specifically named in the BC curriculum, it is possible that a future question could require one of these names in students work on the AP exam.