2016 AB-5/BC-5 Dennis Wilson, Landmark Christian School

Problem Overview:

A funnel with circular cross sections is given and the radius is related to the height by the equation $r = \frac{1}{20}(3 + h^2)$, where $0 \leq h \leq 10$.

Part a:

Students were asked to find the average value of the radius.

Part b:

Students were asked to find the volume of the funnel.

Part c:

Students were told the funnel was filled with a liquid that was draining so that the radius of the surface was decreasing at a rate of $-\frac{1}{5}$ inch per second. They were asked to determine the rate of change of the height of the liquid.

General scoring guidelines for the problem:

Students must use calculus to solve the problems. No points in this problem depended on units. No points were given for bald answers.

Part a: (3 points)

The first point was awarded for an integral with the correct limits of integration and an integrand containing any constant multiple of r, or $3 + h^2$. No differential was required to earn this point but if one was present it needed to be the appropriate differential, dh. Student who had no integral in their work could not earn any of the three points on this part.

Any non-zero constant multiple of $3h + \frac{h^3}{3}$ earned the second point for the antiderivative.

The third and final point was awarded for the correct answer of $\frac{109}{60}$. It was in this point that students earned credit for using the constant $\frac{1}{10}$ to find the average value.

Common errors costing points – Most students who lost points only had the integral and stopped. Some students had dr or dx for differential and earned no points. Some students, frequent enough to notice, used "product integration" which resulted in $\frac{1}{20}h\left(3h+\frac{h^3}{3}\right)$ for the antiderivative. Students used Average Rate of Change instead of average value of the function. Arithmetic simplification of correct results to incorrect values is where the majority of students performing good calculus work lost their points. The number of points lost by these students for trying to perform arithmetic operations was saddening. Not only did these students lose points, but they wasted valuable time in working themselves out of the last point for this part.

Common errors not costing points – Students frequently found the antiderivative to be $3x+\frac{h^3}{3}$. No point was deducted for this error if student recovered with appropriate use of the Fundamental Theorem by substituting 10 for *x* and *h*.

<u>Part b:</u> (3 points)

The first point was awarded for the integrand. The constant of π and limits of integration did not matter for this point as any constant multiple of r^2, or $(3+h^2)^2$ was accepted. R^2 was also accepted as the problem was not case sensitive. No differential was required, but again it must be appropriate if present.

A correct antiderivative which yielded any nonzero constant multiple of $9h+2h^3+\frac{h^5}{5}$ earned the second point. Student should have squared binomial and thus have a trinomial.

The third point was earned for the answer of $\frac{2209\pi}{20}$. Credit for the constant of π and the correct bounds came here.

Common errors – The majority of my AB booklets assumed the shape was a cone and failed to treat the problem as $\int Area\ dh$. Those who did had the integral but the students lost points for either squaring the binomial by just squaring each term or trying to use substitution. Some students used the differential dr instead of dh. Students who made this mistake earned none of the points. Some students who were successful with the integration failed to square the constant of $\frac{1}{20}$ before factoring it out of the integral. Again, many students lost the point for the answer because they simplified a correct numerical expression into an incorrect answer due to arithmetic errors. Responses on my BC booklets were much better for this problem. Some students tried unsuccessfully to use shells to find the volume.

<u>Part c:</u> (3 points)

The first two points were awarded for the chain rule. Various forms of the differential were accepted as long as they showed a clear example of the use of the chain rule, e.g. $\frac{dr}{dt}=\frac{1}{10}h\frac{dh}{dt}$ or $dr=\frac{1}{10}h\ dh$. Students could receive one of the two points with any non-linear constant function multiplied by $\frac{dh}{dt}$. A numeric representation of chain rule, $-\frac{1}{5}=\frac{1}{10}(3)\frac{dh}{dt}$, earned both points. Some students solved the equation explicit for h and then found $\frac{dh}{dt}$.

An answer of $-\frac{2}{3}$ was required to earn the third and final point. Students could not earn the point if $\frac{dh}{dt}=\frac{2}{3}$ is given and then stated that the height is decreasing at a rate of $\frac{2}{3}$. This is because the problem asked for the rate of change of the height. A student who earned one of the two points for chain rule could earn the answer point with a consistent, negative answer. For example, a student who used $\frac{dr}{dt}=\frac{1}{10}(3+2h)\frac{dh}{dt}$ and then gave a consistent answer of $-\frac{7}{6}$ would earn one chain rule point as well as the answer point.

Common Errors – Some students did not use chain rule. Many student improperly applied the power rule to the expression for r or forgot to differentiate the constant 3. Many points were lost when students used $\frac{dr}{dt}=\frac{1}{5}$ instead of the negative value. Various arithmetic errors involving division of fractions were made. Some students recognized that the problem was a related rate and tried to use their knowledge of cones or even right triangles to solve the problem.

The reader continued to read for various copying errors in the polynomial part of r.
Three copying errors were accepted.

$$r=(3+h)^2$$
$$r=3+h^p, \qquad p\geq 3$$
$$r=k+h^2, \qquad k\neq 3$$

If students made one of these three copying error in part a or part b, then they were eligible for the first two points and not the third. In part c, students were eligible for one of the first two points and then the third point if consistent with their copying error. This was the only place in this problem that a student could earn an answer point for an incorrect answer. One frequent copying error that was not accepted was $r=(3-h^2)$.

Linkage Errors:
Students lost points for bringing constants in late if expressions were linked with equal sign.

$\frac{109}{6}\rightarrow\frac{1}{10}\cdot\frac{109}{6}=109/60$ correct

$\frac{109}{6}=\frac{1}{10}\cdot\frac{109}{6}=109/60$ incorrect

Decimal Presentation Errors:
Some students used decimal approximations to work through problem. If a point was deducted for a decimal presentation error, then the student earned immunity from all future decimal representation errors in problem. Decimal errors (from arithmetic) did not earn immunity for decimal representation error.

Observations and recommendations for teachers:

(1) The difference between the average value of a function versus the average rate of change of a function should be emphasized to students.

(2) Students should be required to work many examples of integration where simplification with algebra is necessary. These examples should continue to be worked once substitution is introduced. Students should be trained to consider algebra before substitution.

(3) Students should easily identify a related rates problem when a problem mentions time but the quantities in the problem are not explicitly related to time. Many students seem to only be trained on classical problems like the sliding ladder, length of shadow, or leaky cone. While these examples are great, the exam rarely presents a related rates problem in its entirety. The last time was on the 2002 form b. Students should learn to implicitly differentiate various relationships with respect to time.

(4) While all teachers want their students to have good arithmetic skills, the use of arithmetic for simplification should be deemphasized when reviewing for the exam. Too many good calculus students lose points for arithmetic errors. It is doubtful that these students are good at calculus and poor at arithmetic. On the contrary, good arithmetic students are spending too much time on complex arithmetic, and due to expiring time, are simply making careless mistakes. The reader can easily recognize the unsimplified answer and award the answer point.