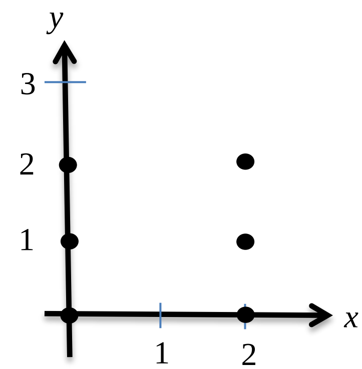
2016 AB4 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$ to consider.

Part a:

Students were given axes similar to that at right and asked to sketch a slope field for the given differential equation at the six points indicated.



Part b:

$y = f(x)$ is given as a particular solution to the differential equation with initial condition $f(2) = 3$. Students were asked to write an equation of the line tangent to the graph of $y = f(x)$ at $x = 2$. This equation then had to be used to approximate the value of $f(2.1)$.

Part c:

Students were asked to find the particular solution to $y = f(x)$ with the initial condition $f(2) = 3$.

Comments on student responses and scoring guidelines:

Part a:

There were two points in this part of the problem. The first was for the zero slopes at the two points on the x-axis and the second point was for the non-zero slopes. Those non-zero slopes had to be shown as negative on the y-axis, positive at the points where $x = 2$, and the slopes shown where $y = 2$ had to be steeper or at least no less than parallel to the slopes where $y = 1$. Students did well on this part of the question, although it was sometimes difficult to read their indications of slopes on the x-axis.

Part b:

The first point of two was for an equation of a tangent line. A correct form of the equation presented with no work shown, such as $y = 9(x-2)+3$ earned the first point. A form of an equation with an incorrect slope indicated as $\frac{dy}{dx} = a \neq 9$ was eligible for the approximation point. Merely stating that $m = a \neq 9$ and using it in an equation earned neither of the two points in this part of the problem. The slope, if incorrect, had to be connected to the differential equation either by using $\frac{dy}{dx} =$ or by showing a numerical calculation that used the given expression for the derivative of y for eligibility for the second point. An equation of a line not containing the point (2, 3) earned neither of the two points. Using $x = 2.1$ and solving for y earned the second point, if the equation and any work presented allowed students eligibility for this second point.

Part c:

A lot hinged on students separating variables correctly. A completely bad separation, such as having differentials in the denominator, resulted in a score of zero of the five points in part (c). An almost correct separation such as $\frac{dy}{y^2} = (x-1)dx$ or $y^2 dy = \frac{dx}{x-1}$ was eligible for the anti-derivative point on the correct side of the equation only, and then only eligible for the "$+C$ and using the initial condition" point for a maximum of 2 out of 5 points. The anti-derivatives given by $-\frac{1}{y} = \ln|x-1|$ earned both the second and third points. An expression such as $\frac{1}{y} = \ln|x-1|$ showed one sign/anti-derivative error and earned one of these two points. An expression such as $-\frac{1}{y} = -\ln|x|$ earned zero of these two points because of two errors on the right side of the equation. To be eligible for the fourth point, students had to have earned a minimum of one of the first three points. For example, the following work earned the first point, one of the anti-derivative points, and the fourth point: $\frac{dy}{y^2} = \frac{dx}{x-1} \rightarrow \ln\left(y^2\right) = \ln(x-1) \rightarrow y^2 = C(x-1) \rightarrow 3^2 = C(2-1)$ but was not eligible for the fifth point, solving for y. Notice in this example that "C" showed up late in the work, but correctly. Unless there was at most one sign error in the anti-derivatives, students were only eligible for the fifth point if the first four points had been earned.

Observations and recommendations for teachers:

(1) Similar problems are found on past AP Calculus exams. See 2008 AB5, 2010 AB6, and 2013 AB6.

(2) Separation of variables is a simple almost "algebraic" task that students should practice and know. Differential equations on the AP Calculus exams amount to nothing more than proper handling of anti-derivatives and an initial condition. This idea sends us all the way back to the first introduction to anti-derivatives that we give our students with a reminder to "do not forget $+C$". Practicing this in situations that involve $\ln|x|$ or such functions as $\sin^{-1}(x)$ is important.

(3) It is a really good idea to have students always use absolute value as in $\int \frac{dx}{x-\pi} = \ln|x-\pi| + C$. Absolute value can be disregarded later if the initial condition forces a positive argument for $\ln(x-\pi)$.

(4) To earn more points after the anti-derivatives are shown, students need only show that the $+C$ is used appropriately and that the initial condition is substituted correctly in the resulting equation. Any arithmetic errors, as in solving for C incorrectly, are penalized in the last, answer, point.