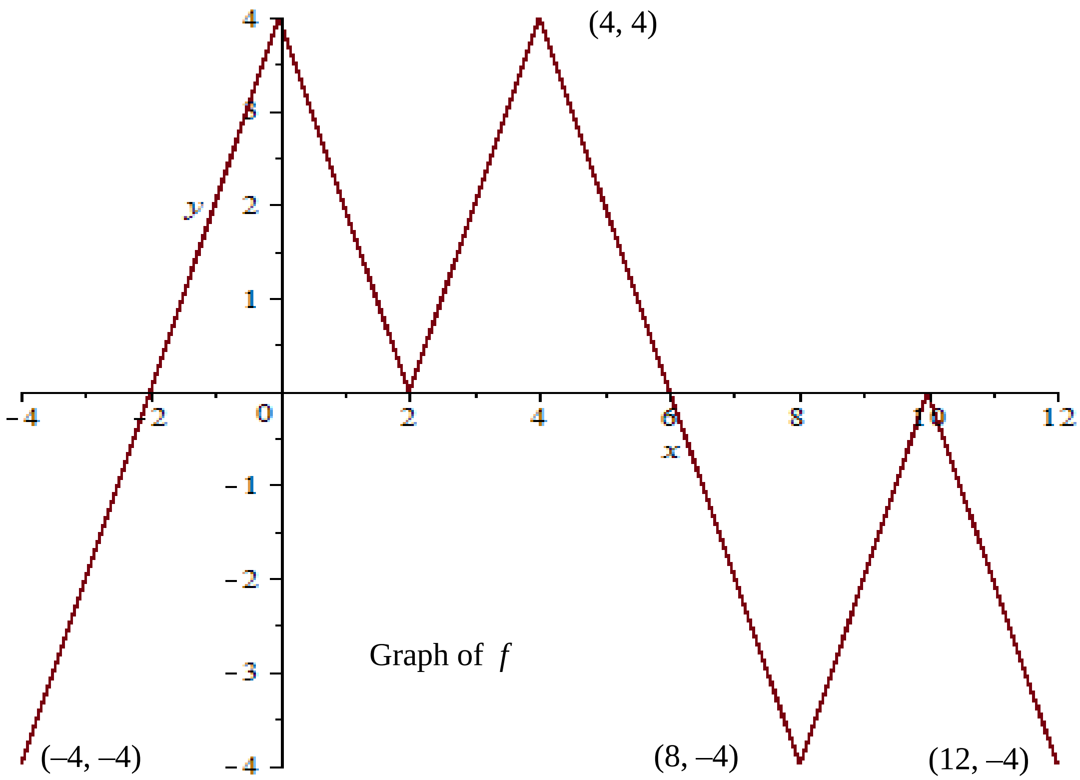
Problem Overview:



Students were given the graph above of a piece-wise linear function f. Students were also given the two facts that g is defined for $-4 \le x \le 12$ and $g(x) = \int_2^x f(t)\,dt$.

Part a:

Students were asked if g has a relative maximum, a relative minimum, or neither at the point where $x = 10$. Students had to justify their answers.

Part b:

Students were asked whether the graph of g has a point of inflection at $x = 4$ and to justify the answer.

Part c:

Students were asked to find the absolute minimum and maximum values of g on $-4 \le x \le 12$ and to justify their answers.

Part d:

Students were asked to find all intervals on $-4 \le x \le 12$ for which $g(x) \le 0$.

Comments on student responses and scoring guidelines:
This problem was worth eight points, the ninth point being for making a connection between f and g'. This connection could be blatant as in $f = g'$, or could appear in other manners such as $f' = g''$ or in a statement such as "since f is the derivative of g" or "the graph shows the derivative of g."
(i) In this problem, readers assumed that "the graph" referred to the only graph given in the stem of the problem, which would be the (graph of) function *f*.
(ii) Student justifications in parts (a), (b) and (c) were not awarded any points unless the connection of f to g' was found somewhere in student work on this question, and the connection point had been awarded.

Part a:

The answer is "neither." A relative min or max has to be justified by noting a change in sign of the derivative of g. There is no sign change in $f = g'$ at the point where $x = 10$. Ambiguous answers such as "the slope does not change at x=10" were not awarded this point (the slope of what?). The justification had to refer to no sign change in the derivative of g or "the graph" or *f*. This part was worth one point.

Part b:

The graph of g does have a point of inflection at the point where $x = 4$ because g'' changes sign at that point. This could be stated correctly in other ways such as "the graph of f changes from increasing to decreasing" or "the slopes of f change sign at $x = 4$" or "f' changes sign at $x = 4$." or "because a relative extremum exists for f' at $x = 4$, f must have a point of concavity." This part was worth one point.

Part c:

The logical argument here is to use the fact that a continuous function will assume absolute max and min values on a closed interval. Considering relative max and min in the interval earned the first point (readers were only looking for consideration of the points where $x = -2$ and 6 because $f = g'$ changes sign at these points). This consideration seen in student work was worth one of the four points in this part of the problem. Consideration of endpoints was worth one point. This was most often seen as students showed values for g at those endpoints. A good number of students listed values for g at all even values of x in the interval. This was fine, usually, to earn the first two points provided all the values of g shown were correct. Unfortunately, many of these students didn't finish part (c) of this question because they did not actually answer the question. Readers will not infer from a table of values given by a student that the max and min values of g have been found. An example of a reason for this would be a student paper which may have all correct values, but stated that a minimum value of g is -4 concluding, therefore, that $-4 < -8$. A response worth all four points would show values of g at all four points under consideration and state the max and min values. The justification followed from listing all important values. Students who did not take this approach to justifying max and min values on a closed interval did not do very well on part (c).

Part d:

The correct intervals are [–4, 2] and [10, 12]. Readers were given detailed instructions for awarding only one of the two points based on student versions of using an open or half-open interval or only a subset of one of the intervals. Students received zero points if values of x outside the correct intervals were included.

Observations and recommendations for teachers:

(1) This problem is a version of finding information about a function given the derivative of that function. Practice with this can begin with a simple example such as $f(x) = 3x - 2$ and stating that this is the derivative of the function g. The first discussion might begin with the open-ended question, "What do we know about the function g?" It is helpful if the concepts of relative extrema, concavity, and points of inflection are already known. What is not yet known in this example given here is an initial condition allowing calculation of specific values of g. Note that this example tells students that f is the derivative of g, information that is not given in this problem (see observation #3 below).

(2) Students on this exam sometimes wrote that they could not find exact values of g because no initial condition had been given. Here is why the definition of a function as a definite integral of another function is important to work with. In fact, there is an initial condition given in this problem in the definition of g. When it is written that $g(x) = \int_2^x f(t)\,dt$, we know that $g(2) = 0$. Thus the point (2, 0) is on the graph of g and we have established an initial condition.

(3) It is important to state why the function f in this problem gives information about g. Correct statements about g or f are not justified if the fact that from $g(x) = \int_2^x f(t)\,dt$, we know that $f = g'$ is not clearly stated by the student. In other words, this problem in giving the graph of f does give the derivative of the function g, but students are not told that and must tell the readers that fact before getting into the basics of this problem (as would be discussed in class in observation #1 above).

(4) Students should know the basic facts that can be determined about a function, given its derivative. If we know that $f = g'$, then statements about f give us information about relative max and min, concavity, and inflection points. These correct statements can be made in a variety of ways. For example, "f is increasing" means that $g'' > 0$, implying that g is concave upward. "The slopes of f are negative" means that $g'' < 0$ implying that g is concave downward. A change in these gives us an inflection point. "f changes from positive to negative" gives us a relative max. No such statements are valid ambiguously. An "ambiguous" statement may include some correct mathematics, but it does not definitively identify the subject of the sentence. Examples are "it is increasing" or "the slopes are negative." We don't know what "it" means because we don't know its antecedent when several functions and their derivatives are under consideration in a problem. In "the slopes are negative" we don't know which slopes are being referenced.