Problem Overview:

Students are given a velocity vector $\left(\cos\left(t^2\right),\, e^{0.5t}\right)$ for a particle in motion along a curve in the xy-plane. The position of the particle is $\left(x(t), y(t)\right)$; and at time $t = 1$, the particle is at the point (3, 5).

4

Part a:

Students were asked to find the x-coordinate of the position of the particle at time $t = 2$.

Part b:

Students were told that for $0 < t < 1$, there is a point on the curve where the line tangent to the curve has a slope of 2. Students were asked to find the time the object is at that point.

Part c:

Students were asked to find the time when the object has a speed of 3.

Part d:

This part asked students to calculate the total distance traveled from time $t = 0$ to time $t = 1$.

Comments on Scoring:
Part a:

To answer this part of the question, students had to use an integral along with the initial condition $x(1) = 3$. The first of three points was for a definite integral, correctly using 2 as in $\int_A^2 \cos t^2$ (students not being penalized for a missing differential). The second point is for "using" the initial condition. This required correct use of both facts $t = 1$ and $x = 3$. Thus $3 + \int_0^2 \cos\left(t^2\right) dt$ earned the integral point but not the point for using the initial condition because of the 0 as the lower limit rather than 1. $\int_1^2 \cos t^2 + 3$ earned neither of the first two points because in this setup, the missing differential creates an ambiguous situation. $\int_1^2 \cos t^2\, dt + 3 = 2.557$ (or 2.556) earned both of the first two points and the third, answer, point as well.

Part b:

Students needed to show use of $\frac{y'(t)}{x'(t)}$ with the given velocity components declared (slope had to be shown in terms of t). Thus the mere presence of $\frac{y'(t)}{x'(t)}$ in the absence of a connection to $\left(\cos\left(t^2\right),\, e^{0.5t}\right)$ did not earn the first of two points in this part of the problem. The statement "$\frac{e^{0.5t}}{\cos\left(t^2\right)}$ and $t = 0.84$" earned only the first point and not the answer point because the equation $\frac{e^{0.5t}}{\cos\left(t^2\right)} = 2$ had to be shown in order to indicate where the value 0.84 came from. In this calculator active problem, showing this equation amounts to "show your work."

Part c:

This part was graded using the same philosophy as in part b. Students had to show the speed in terms of t and if using $\sqrt{(x')^2 + (y')^2}$ had to show the connection to $\left(\cos\left(t^2\right),\, e^{0.5t}\right)$. Again, in order to earn the answer point, students had to show the equation that leads to that answer. In this part, the equation is $\sqrt{\cos^2\left(t^2\right) + e^t} = 3$. There were two points in part c, the second for the answer either $t = 2.196$ or 2.197.

Part d:

The first point was for a definite integral of the speed which could be an error imported from part c provided it was of the form $\sqrt{\text{trig}^2 + \exp^2}$. Importing such an error made the student ineligible for the answer point. The correct form is $\int_0^1 \sqrt{\cos^2\left(t^2\right) + e^t}\, dt$. The answer point was awarded for either $t = 1.595$ or 1.594.

Observations and recommendations for teachers:

(1) Calculating a result, given an initial condition and using a definite integral of a rate of change, is a fundamental concept irrespective of motion of a particle. Underlying this is use of the fundamental theorem given by $\int_c^d f(x)\,dx = A(d) - A(c)$ in the form $A(d) = A(c) + \int_c^d f(x)\,dx$ where A is an antiderivative of f. As discussed above, correct use of a differential is important.

(2) Motion with position given by $(x(t), y(t))$ identifies a curve that is the path the particle follows while in motion. A brief background in curves defined parametrically, including that slope of a line tangent to such a curve is given by $\frac{dy/dt}{dx/dt}$, is a skill that can be discussed without getting into particle motion.

(3) Speed of a motion is given by $|v(t)|$. In two dimensions, v is the vector $\vec{v}(t) = \langle x'(t), y'(t) \rangle$ and we have $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$. Total distance traveled is the definite integral of speed, something that has to be reinforced and practiced in class. $\int_{t_1}^{t_2} |v(t)|\, dt$ has the same meaning in 1 dimension as in 2 (or 3?).

Contrast this with displacement (the **net** change in position) given by $\int_{t_1}^{t_2} v(t)\, dt$.