Calculus II
Area and Arc Length in Polar Coordinates

- Summary of important concepts

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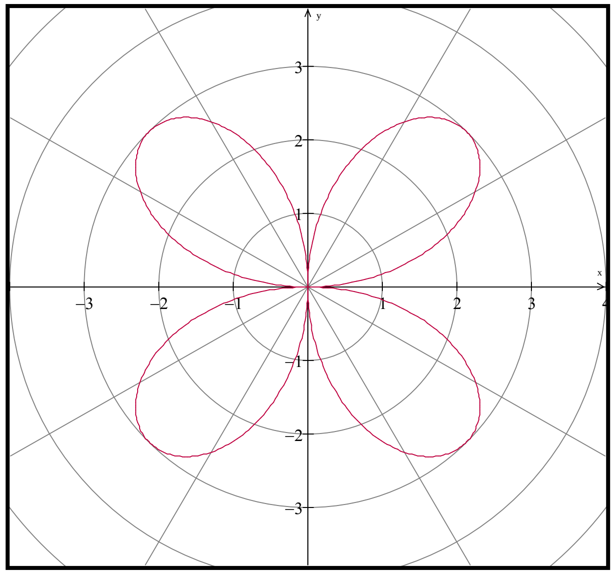
[1] **<u>Area in Polar Coordinates</u>**: A region bounded the Origin and the curve given by $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ has an area given by $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$.

Example 1: The curve defined by $r = 3\sin(2\theta)$ is shown in the graph at right. To find the area bounded by one loop of this curve, we use the definite integral above with lower and upper limits of $\alpha = 0$ and $\beta = \frac{\pi}{2}$. These are the values of θ which form one complete loop. It is important to know those values when finding area.



The curve passes through the Origin when $r = 0 = 3\sin(2\theta)$.
Solving this equation, we get $\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$, etc.

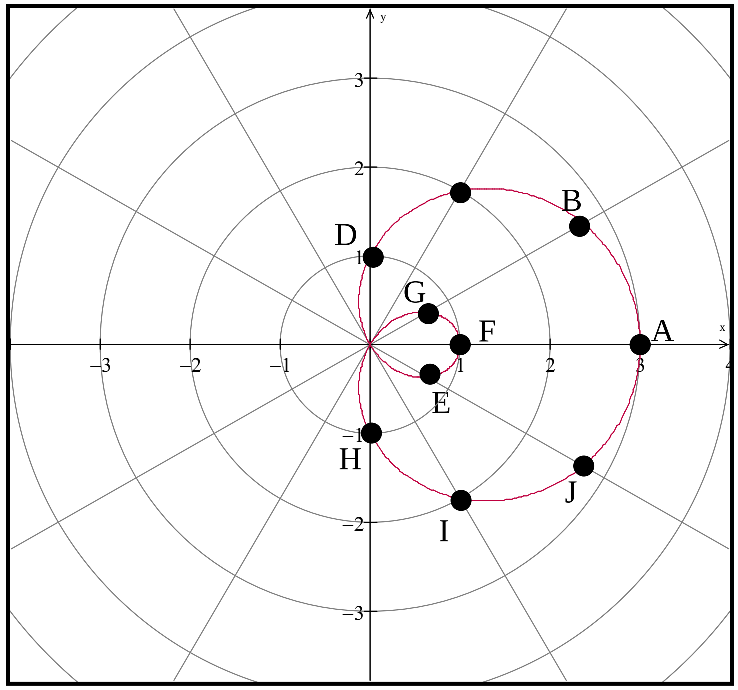
Area = $\frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2(2\theta)\, d\theta$ for one loop.

Area = $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9}{2}\left(1-\cos(4\theta)\right) d\theta = \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{9\pi}{8}$. Thus the area of all four loops is $\frac{9\pi}{2}$.

NOTE: There is a pattern here for $r = \sin(n\theta)$ where $n = 1, 2, 3, \ldots$. Can you find the area for different values of *n* for all the loops? Can you find a general term expressing the area of all the loops in terms of *n*?

Example 2:
Recall the example $r = 2\cos(\theta) + 1$ from notes for section 11.4. Polar coordinates were determined in sketching this curve in the notes for section 11.4, but not points at the Origin. How do we find the area bounded by the inner loop of this graph? The graph is shown below right.

	polar (r, θ)
A	(3, 0)
B	$\left(\sqrt{3}+1, \frac{\pi}{6}\right)$
C	$\left(2, \frac{\pi}{3}\right)$
D	$\left(1, \frac{\pi}{2}\right)$
E	$\left(1-\sqrt{3}, \frac{5\pi}{6}\right)$
F	$(-1, \pi)$
G	$\left(1-\sqrt{3}, \frac{7\pi}{6}\right)$
H	$\left(1, \frac{3\pi}{2}\right)$
I	$\left(2, \frac{5\pi}{3}\right)$
J	$\left(\sqrt{3}+1, \frac{11\pi}{6}\right)$
Origin	$\left(0, \frac{2\pi}{3}\right)$
Origin	$\left(0, \frac{4\pi}{3}\right)$



We must determine the values of θ for which the curve is at the Origin in order to know the limits to use in the definite integral. In the graph above, the bottom of the inner loop begins to form at the **Origin** where $\theta = \frac{2\pi}{3}$ and passes through point E to point F at (1, 0) where $\theta = \pi$. Solve the equation $r = 2\cos(\theta) + 1 = 0$ to get $\theta = \frac{2\pi}{3}$.

We can double the area of the bottom half of this loop to get the area of the entire loop:

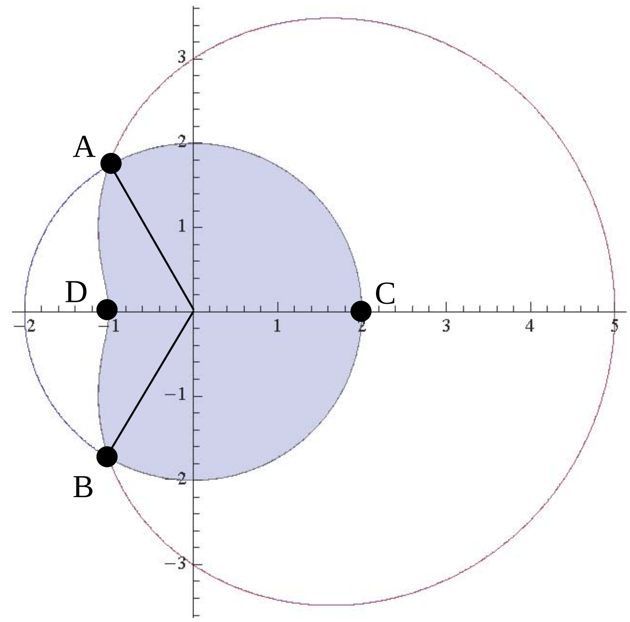
The area is given by $2\left(\frac{1}{2}\right)\int_{\frac{2\pi}{3}}^{\pi} \left(2\cos(\theta)+1\right)^2 \, d\theta = \pi - \frac{3\sqrt{3}}{2}$.

Alternatively: We could solve the equation $r = 2\cos(\theta) + 1 = 0$ for the next value of $\theta = \frac{4\pi}{3}$ and calculate the area of the entire loop rather than doubling the bottom half.

The area is given by $\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left(2\cos(\theta)+1\right)^2 \, d\theta = \pi - \frac{3\sqrt{3}}{2}$.

OR the area is given by doubling the top half of the inner loop: $2\left(\frac{1}{2}\right)\int_{\pi}^{\frac{4\pi}{3}} \left(2\cos(\theta)+1\right)^2 \, d\theta = \pi - \frac{3\sqrt{3}}{2}$

Example 3: Intersecting curves
The curves $r = 2$ and $r = 3 + 2\cos(\theta)$ determine three distinct regions. To find the area of the middle region shaded in the graph at right, notice that $r = 2$ is a circle of radius 2. Part of the shaded region is a part of this circle. How much of the circle is shaded? We must find the intersection points A and B of these two curves. We set the expressions equal and solve:



$r = 2 = 3 + 2\cos(\theta)$

$-1 = 2\cos(\theta)$

$-\frac{1}{2} = \cos(\theta)$

$\theta = \frac{2\pi}{3}$ at A and $\theta = \frac{4\pi}{3}$ at B

These two rays and $r = 2$ define 2/3 of the circle.

The area bounded by $\widehat{\mathrm{ACB}}$ on $r = 2$ and the rays $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ is given by $\frac{2}{3}\pi(2)^2 = \dfrac{8\pi}{3}$.

Alternatively: This same area could be found by subtracting the area bounded by $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ on the left side of the circle from the entire area of the circle: $\pi(2)^2 - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2)^2\, d\theta = \dfrac{8\pi}{3}$

OR directly from $\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (2)^2\, d\theta = \dfrac{8\pi}{3}$.

The area bounded by rays $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ and $r = 3 + 2\cos(\theta)$ on the left side of the region (see point D) is found with our definite integral $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$:

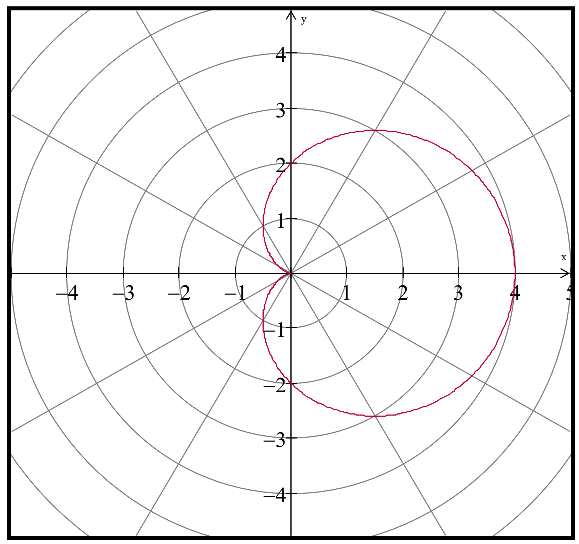
$$\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (3 + 2\cos(\theta))^2\, d\theta = \frac{1}{2}\left(9\theta + 12\sin(\theta) + 2\theta + \sin(2\theta)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \frac{-11\sqrt{3}}{2} + \frac{11\pi}{3}.$$

Therefore, the entire area of the region shaded is given by $\dfrac{-11\sqrt{3}}{2} + \dfrac{11\pi}{3} + \dfrac{8\pi}{3} = \dfrac{-11\sqrt{3}}{2} + \dfrac{19\pi}{3}$.

[2] **<u>Arc length of a curve defined in polar coordinates:</u>** If a curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is traced exactly once, the length of the curve is given by $\displaystyle\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$. This is not usually a simple anti-derivative to evaluate.

Example: 1

Find the entire length of the cardioid given by $r = 2 + 2\cos(\theta)$. This curve is graphed at right and the complete curve is shown for $0 \le \theta \le 2\pi$.



We have $\dfrac{dr}{d\theta} = -2\sin(\theta) \rightarrow \left(\dfrac{dr}{d\theta}\right)^2 = 4\sin^2(\theta)$.

We can calculate this length by finding half for $0 \le \theta \le \pi$ and then doubling.

The arc length is calculated as follows:

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 2\int_0^{\pi} \sqrt{\left(2 + 2\cos(\theta)\right)^2 + 4\sin^2(\theta)}\, d\theta = 2\int_0^{\pi} \sqrt{4\left(1 + \cos(\theta)\right)^2 + 4\sin^2(\theta)}\, d\theta$$

This integral is the same as $4\int_0^{\pi} \sqrt{\left(1 + \cos(\theta)\right)^2 + \sin^2(\theta)}\, d\theta = 4\sqrt{2}\int_0^{\pi} \sqrt{1 + \cos(\theta)}\, d\theta$.

The anti-derivative can be calculated using the identity $\cos^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 + \cos(\theta))$. The final answer is 16.

NOTE: Using the identity $\cos^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 + \cos(\theta))$ leads to the integral of $\cos\left(\frac{\theta}{2}\right)$.

If we try to find the entire arc length using limits of $\theta = 0$ and $\theta = 2\pi$ we note that $\int_0^{2\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 0$.

This is not a valid answer because the length of this curve is clearly NOT 0.

In fact, $\cos^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 + \cos(\theta)) \rightarrow \sqrt{1 + \cos(\theta)} = \sqrt{2}\left|\cos\left(\frac{1}{2}\theta\right)\right|$ NOT $\sqrt{2}\cos\left(\frac{1}{2}\theta\right)$.